Completing the Square

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Seems like we should finish a picture when we say complete the square. Doesn't It?

Remember when we memorized special products back when we were working with polynomials.

One of those $(a + b)^2 = a^2 + 2ab + b^2$, $a^2 + 2ab + b^2$ is called a perfect square because it came from squaring $(a + b)$, Using that pattern, let's look at some special products.

$(x + 3)^2 = x^2 + 6x + 9$
$(x + 5)^2 = x^2 + 10x + 25$
$(x + 10)^2 = x^2 + 20x + 100$
$(x + 4)^2 = x^2 + 8x + 16$

Now, if we studied these long enough, we might see some relationships that really don't just jump out at you. But the relationships are important, so I am going to tell you what they are.

Looking at the linear term and its relation to the constant, (see how important vocabulary is in math), do you see any way to get from the 6 to the 9 in the first special product from 10 to 25 in the second product? Chances are you don't.

$(x + 3)^2 = x^2 + \mathbf{6}x + \mathbf{9}$

If I took half of the 6, and squared it, I'd get 9.

$(x + 5)^2 = x^2 + \mathbf{10}x + \mathbf{25}$

If I took half the 10, and squared it, I would get 25. Getting excited? Looking at;

$(x + 10)^2 = x^2 + 20x + 100$

Taking half of 20 and squaring it gives me 100.

Neat, huh.

What I see is all those perfect squares have that same property. Now if I ask, Is $x^2 + 6x - 11$ a perfect square?

By taking half of 6 and squaring, you don't get 11. Therefore, the pattern of taking ½ and squaring doesn't seem to hold true.

Ok, how can we recognize a perfect square?

That's important, we can use that information we saw from the patterns in perfect squares.

Let's say I was asked to solve the equation:

$x^2 + 6x - 11 = 0$

I can not seem to factor the trinomial. So It looks like I can't solve it.

However, if I could transform that equation into a perfect square, I might be able to solve it.

We know, in order to have a perfect square, the constant term comes from taking half the linear term and squaring it.

So, let's push the (–11) out of the way by adding 11 to both sides of the equation and complete the square.

$$x^2 + 6x - 11 = 0$$

How do we do that? Add 11 to both sides

$$x^2 + 6x = 11$$

Now, let's make a perfect square on the left side of the equation. Take half of 6 and square it, we get 9. This is important, if we add 9 to the left side, we must add 9 to the right side.

$$x^2 + 6x + \underline{\mathbf{9}} = 11 + \underline{\mathbf{9}}$$

Factoring, $(x + 3)^2 = 20$

Now, I can take the square root of both sides.

$$x + 3 = \pm\sqrt{20}$$

$$x = \pm\sqrt{20} - 3 \quad \text{or} \quad x = -3 \pm\sqrt{20}$$

Simplify $\sqrt{20} = \sqrt{4.5} = 2\sqrt{5}$

$$\text{So, } x = -3 \pm 2\sqrt{5}$$

We can use the method of completing the square to solve quadratics we can't factor.

All we are doing is making an equivalent equation by adding a number to both sides of the equation that will make the polynomial a perfect square.

Then we solve the resulting equation.

