Chapter 4 Relations and Functions

Sec. 1 Relations and Functions

Students that have read a menu have experienced working with ordered pairs. Menus are typically written with a food item on the left side of the menu, the cost of the item on the other side as shown:

Hamburger …………$3.50
Pizza...………………..2.00
Sandwich……………..4.00

Menus could have just as well been written horizontally;

Hamburger, $3.50, Pizza, 2.00, Sandwich, 4.00.

But that format (notation) is not as easy to read and could cause confusion. Someone might look at that and think you could buy a $2.00 sandwich. To clarify that so no one gets confused, I might group the food item and its cost by putting parentheses around them:

(Hamburger, $3.50), (Pizza, 2.00), (Sandwich, 4.00)

Those groupings would be called ordered pairs, pairs because there are two items. Ordered because food is listed first, cost is second.

By definition, we have **a relation - any set of ordered pairs**.

Another example of a set of ordered pairs could be described when buying cold drinks. If one cold drink cost $0.50, two drinks would be $1.00, three drinks would be $1.50. I could write those as ordered pairs:

(1, .50), (2, 1.00), (3, 1.50), and so on

From this you would expect the cost to increase by $0.50 for each additional drink. What do you think might happen if one student went to the store and bought 4 drinks for $2.00 and his friend who was right behind him at the counter bought 4 drinks and only paid $1.75?

My guess is the first guy would feel cheated, that it was not right, that this was not working, or this was not *functioning*. The first guy would expect anyone buying four drinks would pay $2.00 - just like he did.

Let's look at the ordered pairs that caused this problem.

(1, .50), (2, 1.00), (3, 1.50), (4, 2.00), (4, 1.75)

The last two ordered pairs highlight the malfunction, one person buying 4 drinks for $2.00, the next person buying 4 drinks for a $1.75.

For this to be fair or functioning correctly, we would expect that anyone buying four drinks would be charged \$2.00. Or more generally, we would expect every person who bought the same number of drinks to be charged the same price. When that occurs, we'd think this is functioning correctly. So let's define a function.

A function is a special relation in which no two different ordered pairs have the same first element. Since the last set of ordered pairs have the same first elements, those ordered pairs would not be classified as a function.

If I asked students how much would 10 cold drinks cost, many might realize the cost would be \$5.00. If I asked them how they got that answer, eventually, with prodding, someone might tell me they multiplied the number of cold drinks by \$0.50. That shortcut can be described by a rule

Cost = \$0.50 x number of cold drinks

$$c = .50n$$

or the way you see written it in your math boob

$$y = .50x \quad \text{or} \quad y = \tfrac{1}{2}x$$

That rule generates more ordered pairs. So if I wanted to know the price of 20 cold drinks, I would substitute 20 for x. The result would be \$10.00. Written as an ordered pair, I would have (20,10)

Let's look at another rule.

EXAMPLE $y = 3x + 2$

If we plug 4 in, we get 14 out, represented by the ordered pair (4, 14)

If we plug 0 in, we get 2 out, represented by (0, 2)

There is an infinite number of numbers I can plug in.

A RELATION is any set of ordered pairs. The set of all first members of the ordered pairs is called the **DOMAIN** of the relation. The second numbers are called the **RANGE** of the relation.

Sometimes we put restrictions on the numbers we plug into a rule, the domain. Those restrictions may be placed on the relation so it fits real world situations.

For example, using our cold drink problem. If each drink costs \$0.50, it would not make sense to find the cost of –2 drinks. You can't buy negative two drinks, so we would put a restriction on the domain. The only numbers we could plug in are positive whole numbers; 0, 1, 2, 3, …

The restriction on the domain also effects the range. If you can only use positive whole numbers for the domain, what values are possible for the range?

We defined a FUNCTION as a special relation in which no two ordered pairs have the same first member.

What that means is that for every member of the domain there is one and only one member of the range. That means if I give you a rule, like $y = 2x - 3$, when I plug in $x = 4$, I get 5 out. Represented by the ordered pair (4, 5). Now if I plug in 4 again, I have to get 5 out. That makes sense, it's expected so just like the rule for buying cold drinks, this is working, this is functioning as expected, it's a function.

Now, you are thinking, big deal, isn't that what we would expect?

Looking at another rule might give us a clue, $x^2 + y^2 = 25$.

Solving that for that y, we get $y^2 = 25 - x^2$

$$y = \pm\sqrt{25 - x^2}$$

Now if we plug in a number like 3, we get two answers, (3, 4) and (3, –4). You can see there is not one and only one member in the range for each member in the domain. Therefore this rule describes a relation that is not a function.

We can look at the graphs of a relations, nothing more than a bunch (set) of ordered pair,s (points) and determine if it's a function.

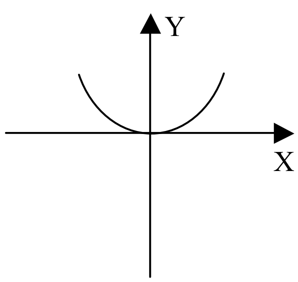
To determine if a graph describes a function, you use what we call the **Vertical Line Test**. That is, you try to draw a vertical line through the graph so it interests the graph in more than one point. If you can do that, then those two ordered pairs have the same first element, but a different second element. Therefore the graph would not describe a function.

If there does not exist any vertical line which crosses the graph of the relation in more than one place, then the relation is s function. That is called the VERTICAL LINE TEST.

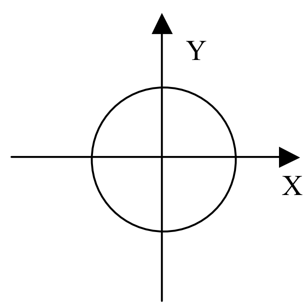
What we try to do is draw a vertical line so it intersects the graph in more than one place. If we can't then we have a graph of a function.

Try these-

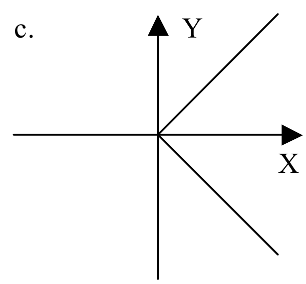
a.



b.



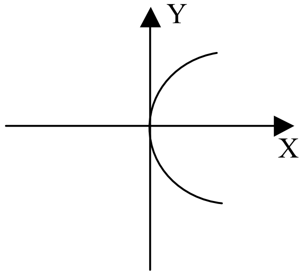
c.



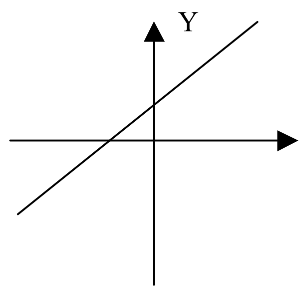
Why does this work? Well if we think about it for a few minutes we can see if there is more than one intersection, then that particular X has two different Y's associated with it.

Label the following as relations or functions.

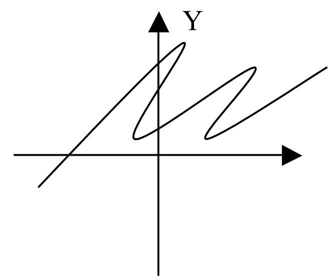
d,



e.



f.



Only a. and e. are functions.

We use functions almost every day in our lives, because we are just living, sometimes we don't think of it as mathematics.

Let's look at a case in point. Let's say you have a cell phone, the phone company charges you $10.00 per month plus $0.05 per minute.

Without a lot of math, you know if you don't use the cell phone that month, you will be charged a flat rate of $10.00. If you speak for one minute, the charge will be $10.05, 2 minutes will be $10.10, 3 minutes $10.15. If you spoke for one hour, you would be charged $13.00.

I could describe that mathematically

$$\text{cost} = \$10.00 + \$0.05\text{minute}$$

$$c = 10 + .05m \quad \text{or} \quad c = .05m + 10$$

or the way you see it written in math class

$$y = .05x + 10$$

Where y represents the cost and x represents the minutes spoken on the phone.

Using fractions, that equation might also look like $y = 1/20\ x + 10$

Since those rules generate ordered pairs, all I need to do is plug in numbers to find the cost of using my phone for any number of minutes.

Using the rule $y = .05x + 10$, I might ask how much it would cost to use the phone if I spoke for ten minutes.

Someone else might ask how much it might be if they used the phone 20 minutes.

Still more people could ask the cost of operating the phone for any number of minutes.

As you can see, to ask these questions a full sentence has to be written, then we plug the desired number of minutes into our rule; $y= .05x + 10$

Rather than wasting all that time and paper writing sentences, it might be nice to develop a shorthand method (notation) to describe the same situation.

Let's go back to $c = .05x + 10$ to describe the cost of using the phone. Remember c represent the cost, x represents the number of minutes.

$$c = .05x + 10$$

Rather than writing sentences to find the cost of speaking for 10 minutes, 20 minutes or any number of minutes , I could write these mathematically.

$c(x) = .05x + 10$	read the value of c at x is .05 + 10
$c(10) = .05(10)x + 10$	read the value of c at 10 is .05x + 10
$c(20) = .05((20) + 10$	read the value of c at 20 is .05x +10
$c(30) = .05(30) + 10$	read the value of c at 30 = .05x + 10

and for x minutes it would be $c(x) = .05x + 10$, read the value of c at x = .05x + 10.

Some like to say c **of** x = .05x + 10

Sec. 2 Combining Functions

Functions have some important properties that we often study later.

If f and g are two functions with a common domain, then the sum of f and g, is defined to be:

$$(f + g)(x) = f(x) + g(x)$$

The difference of f and g is defined by: $(f - g)(x) = f(x) - g(x)$ and the quotient of f and g is defined by $(f/g)(x) = \dfrac{(f)(x)}{g(x)}$ where g(x) cannot be zero.

Let's see what all that means.

If $f(x) = 3x$ and $g(x) = x - 4$,

Now we will plug in a number for f and g, if x = 2, then

$$f(2) = 6,$$
$$g(2) = -2$$

Adding those together, I get $f(x) + g(x) = 6 + (-2)$
$= 4$

If I were to add those functions $f(x) + g(x) = (f + g)(x)$

$$(f + g)(x) = 3x + (x - 4)$$
$$= 4x - 4$$

Now, if I plug 2 into that function, I get 4, just like I did before.

$$(f + g)(x) = 4x - 4$$

What we can see from this example is I can find the value of f(2) and the value of g(2) and add those two results together to find an answer or I could have combined f and g first into one rule, then found (f+g)(2). The answer is the same.

You see that it works. Math = your life.

While you may be wondering why someone might want to even bother with this, later it will help us in our graphing and save us some time computationally. By combining rules as we did with f and g, we can cut our work down considerable – particularly if we had a lot of computations to do.

Sec. 3 Composition of Functions

There are times when we may not want only to add or subtract functions as we just did, but compose a new function because the data we substitute in the first will be used to find the result of a second function.

For instance, let's say you are billed for your cell phone at a rate described by the following function (rule).

$$C = 0.05x + 10$$

In other words the cost of your cell phone is $10.00 per month plus five cents for each minute you speak.

Let's suppose you spoke for twenty minutes, you would be billed $11.00 for the month.

Now, let's say you are taxed at 8% on that amount and that is added to your bill. Well, that's easy enough, I find the cost of the cell phone, take 8% of that number and add that sum to the bill. In our case, 8% of $11.00 is $0.88. So our bill is $11.88.

Now, if I had one thousand customers and I wanted to to find their monthly tax bill. To accomplish that, I would have to find the monthly charge, then take 8%. While that's not hard work, there's two steps of computation that have to be completed.

Wouldn't it be nice if I could find a way of combining those functions into one rule – eliminating one of the computations?

Let's rewrite these two rules using mathematical notation. We'll let f describe the cost of the cell phone as previously described:

$$f(x) = 0.05x + 10$$

And g describe the amount of tax to be paid based upon that bill.

$$g(x) = .08x$$

As we have just done, to find the cost of the cell phone plus tax, I would have to plug into f the number of minutes I spoke, take that result and plug that into g to find the tax, and finally, add those two numbers together.

As you can see, for each customer I have to perform three computations, find f, find g, then find the sum of f and g.

Composition of functions allows me to combine functions when the second function depends upon the value of the first function. As we saw, g, the tax was dependent upon the monthly phone charge – f.

Let's do a quick review before going any further. We introduced functional notation as a shorthand notation to determine values for various data points.

So $h(x) = 4x + 5$ was read as "h at x is equal to 4x + 5"

If I wanted to determine the value of h at 6, I would write $h(6) = 4(6) + 5$ or $h(6) = 29$.

The point I want to drive home is – everywhere I saw an x, I substitute a 6.

$$h(x) = 4x + 5$$

$$h(6) = 4(6) + 5$$

To compose a new function that will allow me to combine these two rules into one, I need to use this very same strategy of substitution.

So, let's go back to our phone problem, we have

$$f(x) = 0.05x + 10$$

$$g(x) = .08x$$

If all I want to do is find the tax described by g, then I'm going to plug in f(x) everywhere I see an x in g.

$g(x) = .08x$	Given
$g\{f(x)\} = .08(f(x))$	Substituting f(x) for x
$g\{f(x)\} = .08\ (.05x + 10)$	Substituting .05x + 10 for f(x)
$g\{f(x)\} = .004x + .80$	Simplifying (D-Property)

This new function allows me to find the tax without first finding the monthly phone charge. So, to find the tax on using the phone 20 minutes, I merely plug 20 into my new formula.

$$g\{f(20)\} = .004(20) + .80$$

$$= ,08 + .80$$

$$= \$0.88$$

Now, if I wanted to use one rule to find the monthly cost plus tax, I would add the tax rule (.004x + .80) that we just found to the f, the monthly cost of the phone. Let's call that rule T.

$$T(x) = ,004x + .80 + f(x)$$

$$T(x) = .004x + .80 + (.05x + 10)$$

$$T(x) = .054x + 10.80$$

What this allows us to do is use one rule to find the cost of using a cell phone plus tax. Otherwise, you'd have to find the monthly cost of the phone, find the tax based on that cost, then add those two amounts together. While that would not be much work if you only had to calculate a few bills, it would be a real pain in the neck if you had to calculate 1000 phone bills.

Composing the new rule might take a couple of minutes up front, but it will save a lot of time in the long run. Let's see how the new rule works for a monthly bill when you spoke for 20 minutes.

$$T(x) = .054x + 10.80$$

$$T(20) = .054(20) + 10.80$$

$$T(20) = 1.08 + 10.80$$

$$T(20) = 11.88$$

Try the following compositions.

1. If $f(x) = 2x + 5$ and $g(x) = 3x + 1$, find $f\{g(x)\}$

2. From problem #1, find $g\{f(x)\}$

3. If $h(x) = x^2$ and $p(x) = x + 3$, find $h\{p(x)\}$

4. From problem #3, find $p\{h(x)\}$