Chapter 6

Applying Congruent Triangles

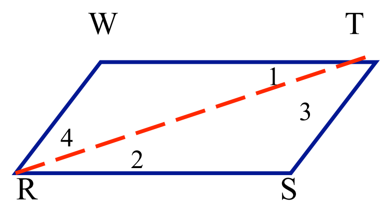
A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

In this chapter, we will use our knowledge of congruent triangles to find other relationships. The most common way of finding these relationships is turning something that we are not familiar with into something we have more knowledge and understanding – triangles. In our first theorem, we are given a parallelogram. The only thing we know about a parallelogram comes from the definition. However, if I can introduce triangles into the picture, I have a better chance of finding relationships.

Theorem A diagonal of a ||ogram separates the ||ogram into 2 ≅ Δ's

Given: ▱RSTW

Provw: ΔRST ≅ ΔTWR



	Statements	Reasons
1.	RSTW is a ||ogram	Given
2.	$\overline{RS} \parallel \overline{WT}$	Def - ||ogram
3.	∠1 ≅ ∠2	2 || lines cut by t, alt int∠'s ≅
4.	$\overline{RT} \cong \overline{RT}$	Reflexive
5.	$\overline{RW} \parallel \overline{ST}$	Def - ||ogram
6.	∠3 ≅ ∠4	2 || lines cut by t, alt int∠'s ≅
7.	ΔRST ≅ ΔTWR	ASA

Knowing the diagonal separates a parallelogram into 2 congruent triangles suggests some more relationships.

Looking at the congruent triangles formed by the diagonal, we can see other relationships using the cpctc.

Theorem The opposite sides of a parallelogram are congruent.

Theorem The opposite angles of a parallelogram are congruent.

Both of those can be proven by adding CPCTC to the last proof.

Many students mistakenly think the opposite sides of a parallelogram are equal by definition. That's not true, the definition states the opposites sides are parallel. This theorem allows us to show they are also equal or congruent.

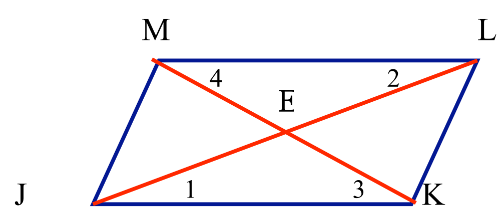
The idea of using cpctc after proving triangles congruent by SSS, SAS, ASA, and ASA will allow us to find many more relationships in geometry.

Let's look at some more proofs.

Theorem The diagonals of a ||ogram bisect each other.

Given: ▱JKLM

Prove: $\overline{JE} \cong \overline{LE}$; $\overline{KE} \cong \overline{ME}$



	Statements	Reasons
1.	JKLM is ||ogram	Given
2.	$\overline{JK} \parallel \overline{ML}$	Def - ||ogram
3.	$\angle 1 \cong \angle 2$	2 || lines cut by t, alt int∠'s
4.	$\overline{JK} \cong \overline{ML}$	opposite sides ||ogram ≅
5.	$\angle 3 \cong \angle 4$	2 || lines cut by t, alt int∠'s
6.	$\Delta JEK \cong \Delta LEM$	ASA
7.	$\overline{JE} \cong \overline{LE}$ $KE \cong ME$	cpctc

Notice to prove this theorem, I first drew the parallelogram, then I drew in the diagonals. In order to prove triangles congruent, I had to add angles to the picture, so I labeled angles 1, 2, 3, and 4 and developed the relationships based upon my previous knowledge of geometry.

Drawing and labeling the information given to you is important. It is also important to label other information in your picture from your previous knowledge of geometry. You need to remember and be able to visualize your definitions, postulates and theorems.

Sometimes we can be given information about a quadrilateral and we could develop more information if we knew the quadrilateral was a parallelogram. So being able to show a quadrilateral is a parallelogram can be important to us. The way that is done is by having the quadrilateral satisfy the definition of a parallelogram.

Theorem If two sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

To prove that theorem, you'd draw a picture of a quadrilateral, construct a diagonal, show two triangles congruent, name he corresponding parts, then you will find the other lines parallel. That satisfies the definition.

Theorem If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Theorem If the diagonals of a quadrilateral bisect each other, the the quadrilateral is a parallelogram.

Using my triangle congruence knowledge, I can further develop my knowledge of parallelograms, rectangles and rhombi.

Let's begin by defining a rectangle. A **rectangle** is a parallelogram with one right angle.

Theorem The diagonals of a rectangle are congruent.

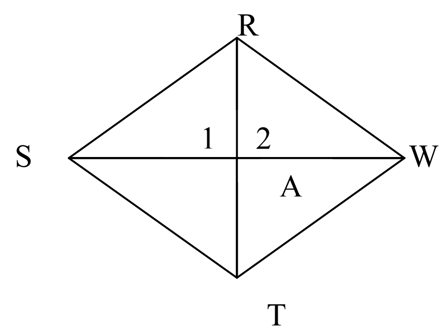
Draw the picture and see if you can see how to go about proving that theorem.

Another special kind of parallelogram is a rhombus. The plural of rhombus is rhombi. A rhombus is a parallelogram with congruent sides. The next two theorems apply to rhombi and not necessarily to all other parallelograms

Theorem: The diagonals of a rhombus are $\perp$

Given: Rhombus RSTW

Prove: $\overline{RT} \perp \overline{SW}$



Statements		Reasons
1.	RSTW – Rhombus	Given
2.	$\overline{RS} \cong \overline{RW}$	Def – Rhombus
3.	SA ≅ WA	Diagonals ||ogram bisect each other
4.	$\overline{RA} = \overline{RA}$	Reflexive
5.	ΔRSA ≅ ΔRWA	SSS
6.	∠1 = ∠2	cpctc
7.	$\overline{RT} \perp \overline{SW}$	2 intersecting lines form ≅ adj ∠'s

The following theorem follows from this proof and the theorem that states if two sides of a triangle are congruent, the angles opposite those sides are congruent.

Theorem Each diagonal of a rhombus bisects a pair of opposite angles.

As you can see, the congruence theorems allow us to determine other mathematical relationships by going one step further and using cpctc.

There are other strategies we can use to show other relationships. We can combine our knowledge of algebra and demonstrate other relationships.

A **trapezoid** is a quadrilateral with exactly one pair of parallel lines. The parallel lines are called the **bases** and the non-parallel lines are called **legs**.

The line segment that joins the midpoints of the legs is called the **median**.

The next two theorems would be a little tough to prove using the current method. Proofs by the methods used in coordinate geometry will make these a lot easier to prove.

Coordinate Geometry

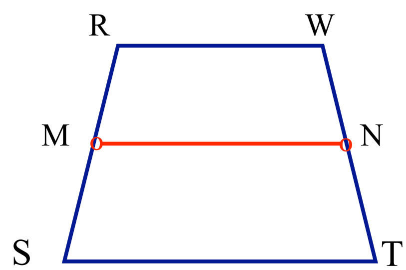
We've been using deductive reasoning in a t-proof to prove theorems thus far. Sometimes, theorems might be better proven using coordinate geometry. In a nutshell, coordinate geometry allows us to express our knowledge of geometry using algebra.

We are going to look at a couple of proofs using coordinate geometry.

Theorem The median of a trapezoid is || to the bases and is equal to half the sum of the bases.

Given: RSTW is a trap

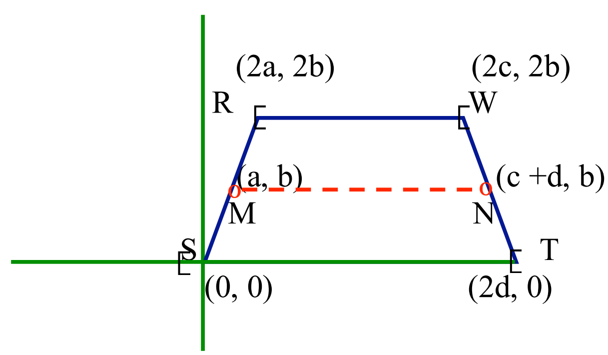
Prove: MN || ST
MN || RW
$MN = \frac{1}{2}(ST + RW)$



In this theorem, we are required to prove two things; first lines are parallel and second a mathematical relationship.

From the geometry we have already learned, we know how to show lines are parallel by looking at angles. In your algebra class, you learned that parallel lines have the same slope

Place trapezoid on coordinate axes, label points, carefully keeping relationships. Find slopes, || lines have = slopes. Find distances. That sounds easy enough, but it takes a little extra thinking to label things so the arithmetic does not get in the way of what we are trying to prove.



I have very conveniently placed the trapezoid on the coordinate axis having coordinates (0, 0). Now, I could have labeled the coordinates of R as (a, b). The reason I did not do that was because I know I have to find the midpoint of RS and that would have led to a fraction. So I got a little tricky, I labeled R as (2a, 2b). That way the midpoint M is (a, b).

Using the same type of logic, I will label W as (2c, 2b) and T as (2d, 0).

The slope is the change in y over the change is x, so the slope of MN = (b–b)/(c+d–a) or 0.

The slope of $\overline{RW}$ is (2b–2b)/(2c–2a) or 0.

The slope of $\overline{ST}$ is (0–0)/(2d–0) or 0.

Since $\overline{MN}$, $\overline{RW}$, and $\overline{ST}$ have the same slope, those lines are all ||. Or more precisely

MN || ST and MN || RW

We have shown the lines are parallel. Now we have to show the median is half the sum of the bases. Since these are all horizontal lines, all I have to do is subtract to find their distances.

MN = c + d – a

RW = 2c – 2a
ST = 2d

RW + ST = (2c – 2a) + 2d
= 2(c –a + d)

From the algebra we can see that RW + ST is twice MN. Another way to say that is MN is half of RW + ST

$$\mathbf{MN = \frac{1}{2}(STW + RW)}$$

Now we have shown both parts, the lines being parallel and the median being half te sum of the bases.

I can not stress enough the importance of setting this up by labeling points conveniently.

We have proven another theorem, but this time we used coordinate geometry. As you become more comfortable with t-proofs and coordinate geometry, you will have to decide which method to use.

When you are not able to prove a theorem using one method, you now have another way at getting at the proof.

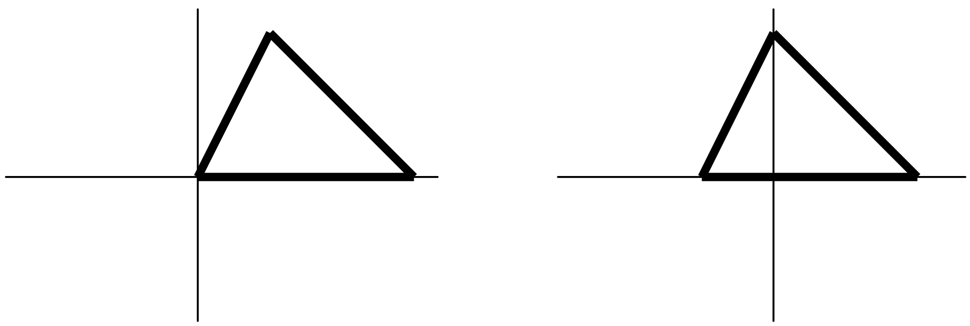
Let's look at another theorem and prove it using coordinate geometry.

Theorem
The segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.

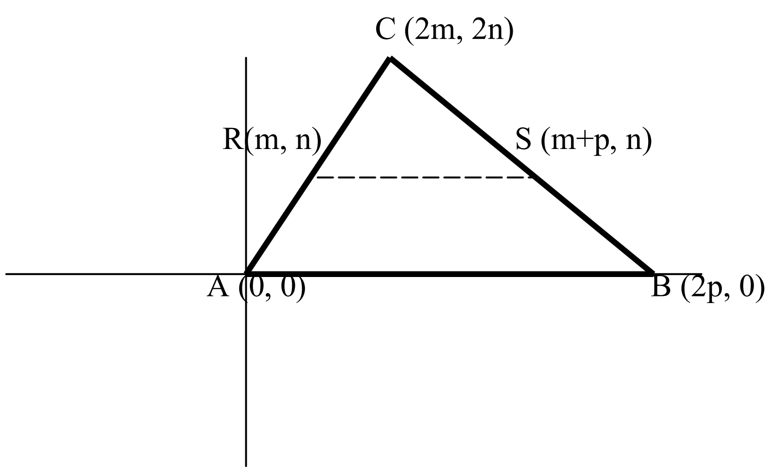
The last proof might suggest to show lines are parallel in coordinate geometry, we need to show they have the same slope.

We also found distances in the last proof, knowing how to find that will help us find the distances in this problem.

Let's look how I positioned the triangles on the coordinate axes. Which positioning do you think might help us with our proof.



Since we are going to be looking for distances and midpoints again, using the origin comes in handy because the coordinates are (0, 0). Let's label the diagram using the positioning on the left.



By labeling C (2m, 2n), the midpoint R is easy to find for $\overline{AC}$. The same is true finding the midpoint S for BC.

The slope of $\overline{RS}$ and $\overline{AB}$ are zero. Since they have the same slope, the lines are parallel.

Let's find the distances -

$$\begin{aligned} RS &= (m + p) - m \\ &= p \end{aligned}$$

$$
\begin{aligned}
AB &= 2p - 0 \\
&= 2p
\end{aligned}
$$

AB is twice RS, that means that RS is half AB.

Mathematically, we have $RS = \frac{1}{2} AB$