Conic Sections

Circles, Parabolas, Ellipses & Hyperbolas

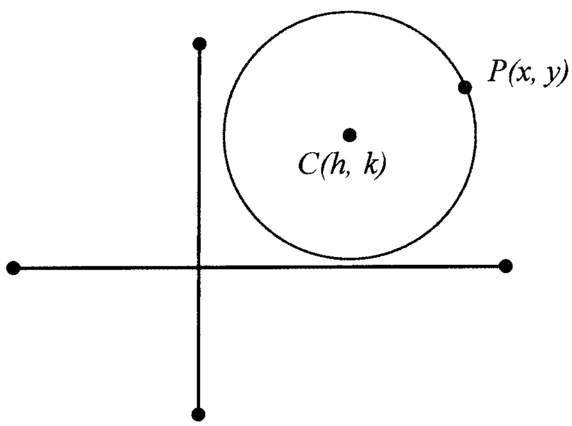
The formulas for the conic sections are derived by using the *distance formula*, which was derived from the Pythagorean Theorem. If you know the distance formula and how each of the conic sections is defined, then deriving their formulas becomes simple. Simplifying the algebraic equations; squaring, combining like terms, factoring, and substituting is all it takes to be successful.

Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Circles

A circle is a set of points P in a plane that are equidistant from a fixed point, called the center.



Let the distance d(C,P) be the radius r of the circle, the center C(h,k) and P(x,y) be a point on the circle. Substituting those into the distance formula, we have:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

Squaring both sides $\quad r^2 = (x-h)^2 + (y-k)^2$

or

$$(x-h)^2 + (y-k)^2 = r^2$$

That is an equation of a circle with center (h,k) and radius r.

Example 1 Find an equation of a circle with center (5, –2) and radius 4.

Using the equation of a circle we just found, we have

$$(x-h)^2+(y-k)^2=r^2$$

Substitute (5, –2) for (h, k) and 4 for r; $(x-5)^2+(y-(-2))^2=4^2$

$$(x-5)^2+(y+2)^2=4^2$$

My guess is most of us could do that more quickly in our head.

Example 2 Find the center and radius of a circle described by $(x+3)^2+(y-1)^2=6^2$

Using the equation of a circle, the center is at (–3, 1) and the radius is 6.

Notice how the signs change!

I could try to make these problems more difficult, but I can't. I can, however, make them longer.

Example 3 Find an equation of a circle with C(2, 6) that passes through P(1,0).

In order to find an equation of a circle, I need to know the center and the radius. The center was given to me, the radius was not. If I can find the distance from the center of the circle to a point P on the circle, then I will know the radius.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$d=\sqrt{(2-1)^2+(6-0)^2}$$
$$d=\sqrt{1^2+6^2}$$
$$d=\sqrt{37}$$

If $d=\sqrt{37}$, then $r=\sqrt{37}$. That means $r^2=37$

Now we know the center and radius, we substitute those numbers in just like we did the last example.

$$(x-2)^2+(y-6)^2=(\sqrt{37})^2$$

Those problems are easy enough if you know the equation of a circle written in **General Form** - $(x-h)^2+(y-k)^2=r^2$. But what if those binomials were expanded. Let's look at the equation in Example 2 and expand it.

$$(x+3)^2+(y-1)^2=6^2 \rightarrow x^2+6x+9+y^2-2y+1=36$$
$$x^2+y^2+6x-2y+10=36$$
$$x^2+y^2+6x-2y=26$$

This is an equation of a circle in **Standard Form**; $Ax^2+By^2+Dx+Ey+F=0$. Notice, in our example, the coefficients of the squared terms are equal. That is A = B. That's important! If A = B, then we have a circle.

So, if we were given the equation, $x^2+y^2+6x-2y=26$, would we know that is a circle. We also know that it is equivalent to $(x+3)^2+(y-1)^2=6^2$ because of the expansion we just performed.

So the question then is, can we re-write $x^2+y^2+6x-2y=26$ in General Form? The answer is yes by *Completing the Square*.

Example 4 Write $x^2+y^2+6x-2y=26$ in General Form and find the center of the circle and its radius.

First group the x's and y's together and leave a space for completing the squares.

$$x^2+6x+\_\_ \quad +y^2-2y+\_\_ \quad =26$$

To complete the square, remember you take half the linear term and square. Be sure to add those amounts to BOTH sides of the equation.

$$x^2+6x+\underline{9}+y^2-2y+\underline{1}=26+\underline{9}+\underline{1}$$
$$x^2+6x+9+y^2-2y+1=36$$
$$(x+3)^2+(y-1)^2=6^2$$

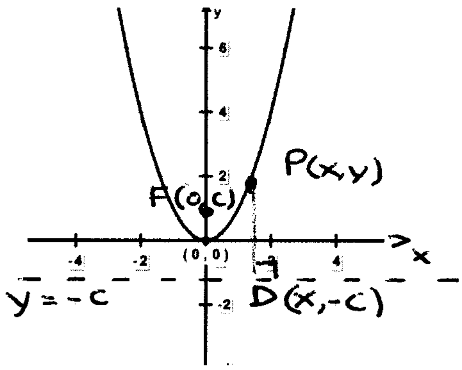
This is a circle with radius 6 and center at (–3, 1) written in General Form.

Parabolas

A parabola is a set of points P whose distance from a fixed point, called the *focus*, is equal to the perpendicular distance from P to a line, called the *directrix*.

Importantly, we see another curve being defined by distances – we need to know the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



By the definition of a parabola, we know FP = PD. Substituting those coordinates into the distance formula, we have

$$\sqrt{(x-0)^2 + (y-c)^2} = \sqrt{(x-x)^2 + (y+c)^2}$$

Squaring, $$x^2 + (y - c)^2 = 0^2 + (y + c)^2$$

Expanding $$x^2 + y^2 - 2yc + c^2 = y^2 + 2yc + c^2$$

Subtracting c^2 & y^2 $$x^2 - 2yc = 2yc$$

$$x^2 = 4yc$$

$$\frac{1}{4c}x^2 = y$$

$$y = \frac{1}{4c}x^2$$

This is an equation of a parabola with vertex at the origin and *c* being the distance between the Focus and the origin and the origin and the directrix.

Mathematically, we write {(x, y)/ $y = \frac{1}{4c}x^2$ } is the graph of a parabola with focus F(0, C) and directrix with equation y = –c.

We can modify this equation and move the vertex from the center to some other point V(h, k) using the formula

$$y - k = \frac{1}{4c}(x - h)^2$$

V(h, k) is the vertex, c is still the distance from the focus to the vertex and the vertex to the directrix. Doing the math, the equation of line representing the directrix is $y = k - c$.

Example 1 Find the vertex, focus and the equation of the directrix for

$$y - 5 = \frac{1}{12}(x - 2)^2$$

The vertex is at (2, 5), to find c, set $\frac{1}{4c}$ equal to the coefficient of the squared term - $\frac{1}{12}$. $\frac{1}{4c} = \frac{1}{12}$, therefore c = 3.

To find the focus, I add 3 to the y-coordinate of the vertex. So F(2, 8). The equation of the directrix is an equation of a line 3 down from the vertex; y = 2.

Example 2 Find an equation of a parabola with vertex (2, 3) and focus (2, 5).

To find the equation of a parabola, we need to know the vertex, which has been given to us, and the distance (c) from the vertex to either the focus or directrix.

$$y - 3 = \frac{1}{4c}(x - 2)^2$$

The distance from the V(2, 3) and F(2, 5) is 2, so c = 2. Substituting c = 2 into the equation, we have

$$y - 2 = \frac{1}{8}(x - 3)^2$$

Before we move on, let's make sure we understand what we have. A parabola has been defined in terms of distances - from the focus and the directrix. Parabolas can open up or down, or sideways – left or right. Graphs where the x's are squared either open up or down. We have been looking at parabolas that have been opening up. That occurs when $\frac{1}{4c}$ is positive and the x's are squared. If $\frac{1}{4c}$ was negative and the x's were squared, the graph would open downward. So I can tell by inspection if the graph is going to open up, down, or sideways just by looking at what terms are squared and the coefficient in from of the squared term.

Now, let's look at another equation of a parabola,

Example 3 Find the vertex, focus, and directrix of $y - 2 = 4(x - 3)^2$

By inspection, we know this is a parabola whose vertex is at (3, 2) and since $4 > 0$, it opens up. To find c, I set $\frac{1}{4c}$ **= 4.** Solving, we have $c = 1/16$.

The focus would be located at $(3,\ 2\frac{1}{16})$ and the equation of the directrix would be

$$y = 1\frac{15}{16}$$

I got those answers by adding 1/16 to the y-coordinate of the vertex and subtracting 1/16 from the y-coordinate of the vertex.

What would the equation in example 3 look like if I expanded the binomial?

$$y - 2 = 4(x - 3)^2 \rightarrow y - 2 = 4(x^2 - 6x + 9)$$

$$y - 2 = 4x^2 - 24x + 36$$

$$y = 4x^2 - 24x + 38$$

Note well! In this equation, only one of the variables has been squared. When this happens, we know the graph will be a parabola. To find the vertex, we will write it in parabola form.

Example 4 Find the vertex, focus, and directrix of $y = 3 - 6x - x^2$

Since only one of the variables is squared, we know this is a parabola and since the coefficient of the squared term is negative, we also know it opens downward. So, just like we did when working with circles, we will *complete the square.*

$$y - 3 = -x^2 - 6x$$

$$y - 3 + \_\_ = -1(x^2 + 6x + \_\_)$$

$$y - 3 - \underline{9} = -1(x^2 + 6x + \underline{9})$$

Caution!!! When I *completed the square*, I added 9 in the parentheses, why did I subtract 9 on the other side of the equation? Because that +9 is being multiplied by a negative one. So I really added a negative 9.

$$y - 12 = -1(x + 3)^2$$

The vertex is located at (–3, 12). $\frac{1}{4c} = -1$, so $c = -\frac{1}{4}$.

Get a visual, we have a parabola that opens downward. Find the vertex and add c to find the focus. The focus F is located at (–3, $11\frac{3}{4}$). The directrix is found on the other side of the vertex, so the equation of the directrix is

$$y = 12\frac{1}{4}$$

We now have the graphs of two quadratic equations that are based on the distance formula. If both x's and y's are squared and the coefficients of the squared terms are equal, the graph will be a circle. In a circle, you should be able to find the center and radius. If a quadratic equations has only one of the variables is squared, the graph will be a parabola. You should be able to find the vertex, focus, and directrix as well as sketch the graph.

Ellipses

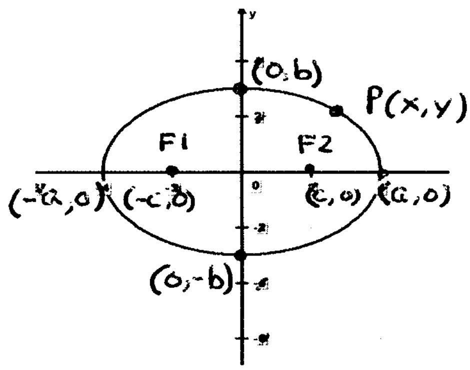
Ellipse is a set of points P in a plane for each of which the sum of the distances from two fixed points, called the foci, is a constant, 2a.

The distances from these two fixed points F1(–c,0) and F2(c, 0) to a point P on the curve are called focal radii of P. The point O bisecting $\overline{F_1F_2}$ is called the center of the ellipse.

The transverse axis has length 2a, the conjugate axis has length 2b.

Again we have a curve being primarily defined by the distance formula.

Distance Formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



In an ellipse, 2a and 2b represent the lengths of the major and minor axes, respectively. The major axis is longer and contains the foci. In an ellipse, $a > b$.

Using the diagram and the Pythagorean Theorem, we have $b^2 = a^2 - c^2$. Use the diagram to see the Pythagorean relationship.

The sum of the focal radii is 2a. By definition; $d(P, F_1) + d(P, F_2) = 2a$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Isolate the radical $\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$

Squaring $(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$

Expanding $x^2 + 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2xc + c^2 + y^2$

Subtracting x^2, y^2, c^2 $2xc = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2xc$

Subtracting 2xc $0 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 4xc$

Divide by 4	$0 = a^2 - a\sqrt{(x-c)^2 + y^2} - xc$
Isolate the radical	$a\sqrt{(x-c)^2 + y^2} = a^2 - xc$
Squaring	$a^2\{(x-c)^2 + y^2\} = a^4 - 2a^2xc + x^2c^2$
Expanding	$a^2\{x^2 - 2xc + c^2 + y2\} = a^4 - 2a^2xc + x^2c^2$
Multiply by a^2	$a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2 = a^4 - 2a^2xc + x^2c^2$
Add $2a^2xc$	$a^2x^2 + a^2c^2 + a^2y^2 = a^4 + x^2c^2$
Subtract a^2c^2, Add x^2c^2	$a^2x^2 + a^2y^2 - x^2c^2 = a^4 - a^2c^2$
Commutative Prop	$a^2x^2 - x^2c^2 + a^2y^2 = a^4 - a^2c^2$
Factor	$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$
Substitute $b^2 = a^2 - c^2$	$x^2b^2 + a^2y^2 = a^2b^2$
Divide a^2b^2	$\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is an equation of an ellipse with center at the origin with x-intercepts a and $-a$ and y-intercepts b and $-b$.

We can move the center as we did with the circle by using ~

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

(h, k) is now the center, the a and b represent the distances from the center (h, k) on the major and minor axes, respectively.

Example 1 Find the center, foci, and x and y intercepts: $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Rewriting, we have $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$

$b^2 = a^2 - c^2$,	center is at (0, 0); $a = 4$, $b = 3$
$3^2 = 4^2 - c^2$	x-intercepts are (4, 0) and (–4, 0)

$9 = 16 - c^2$	y-intercepts are (0, 3) and (0, –3)
$c^2 = 7$	foci are ($\sqrt{7}$, 0) and ($-\sqrt{7}$, 0)
$c = \sqrt{7}$	

Example 2 Find the equation of an ellipse if the length of the minor axis is 6 and the foci are at (4, 0) and (–4, 0).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The foci are on the x-axis , so the x-axis is the major axis and c = 4. The length of the minor axis is 6, so b = 3.

Finding a using $b^2 = a^2 - c^2$, we have

$$3^2 = a^2 - 4^2$$

$$9 = a^2 - 16$$

$$25 = a^2$$

$$5 = a$$

Substituting, $$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

Now, let's look at an equivalent equation by multiplying both sides of $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ by the common denominator, 25 x 9. That gives us the following equation.

$$9x^2 + 25y^2 = 225$$

Notice, when we have a quadratic equation with both x's and y's squared, their coefficients are not equal, but have the same sign, the equation is an ellipse.

In a quadratic equation, if only one variable is squared, the graph is a parabola.

if two variables are squared and the coefficients are the same, then it is a circle.

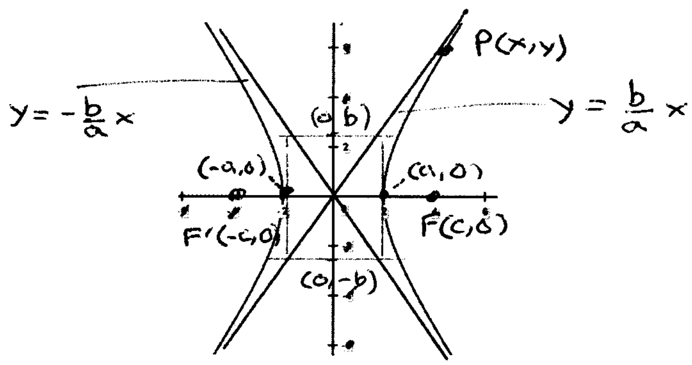
if two variables are squared and the coefficients are not equal but have the same sign, then it is an ellipse.

Hyperbolas

Hyperbola is the set of points in a plane such that for each point, the absolute value of their difference of its distances, called the focal radii, from two fixed points, called the foci, is a constant, 2a.

We have seen previously that the circle, parabola and ellipse were based on the distance formula. Now, we see again, another curve, the hyperbola is based on the distance formula. All we have to do to find the equation of a hyperbola is use the definition of a hyperbola using the distance formula and manipulate the equation.

Distance Formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



By definition; d(P, F) – d(P, F') = 2a. Using that definition, the diagram and $b^2 = c^2 - a^2$

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a$$

Isolate the radical $\sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}$

Square $(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$

Expand binomials $x^2 - 2xc + c^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2$

Subtract x^2, y^2, c^2 $-2xc = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + 2xc$

Subtract 2xc $-4xc = 4a^2 + 4a\sqrt{(x+c)^2 + y^2}$

Divide by 4 $-xc = a^2 + a\sqrt{(x+c)^2 + y^2}$

Isolate the radical $-a^2 - xc = a\sqrt{(x+c)^2 + y^2}$

Square $a^4 + 2a^2xc + x^2c^2 = a^2\{(x+c)^2 + y^2\}$

Expand binomial $a^4 + 2a^2xc + x^2c^2 = a^2(x^2 + 2xc + c^2 + y^2)$

Distribute a^2	$a^4 + 2a^2xc + x^2c^2 = a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$
Subtract $2a^2xc$	$a^4 + x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2$
Subtract a^2c^2, x^2c^2	$a^4 - a^2c^2 = a^2x^2 - x^2c^2 + a^2y^2$
Factor	$a^2(a^2 - c^2) = x^2(a^2 - c^2) + a^2y^2$
Substitute $b^2 = c^2 - a^2$	$a^2(-b^2) = x^2(-b^2) + a^2y^2$
Divide by $-a^2b^2$	$\frac{-a^2b^2}{-a^2b^2} = \frac{-x^2b^2}{-a^2b^2} + \frac{a^2y^2}{-a^2b^2}$

$$\mathbf{1} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad \textbf{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \mathbf{1}$$

This is an equation of a hyperbola that is symmetric to the y-axis – opens sideways. This is an equation of a hyperbola whose center is at the origin with x-intercepts *a* and *–a.* There are no y-intercepts. Remember, y-intercepts occur when x = 0. When x = 0, we end up with a square equaling a negative number. That's not going to happen under the set of Real Numbers.

The equation of a hyperbola that is symmetric to the x-axis – opens sideways is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Notice the a^2 is under the y^2 – the positive quadratic!

In general, to sketch a hyperbola, find the values of *a* and *b*, using $b^2 = c^2 - a^2$. Then determine the vertices by looking for the positive quadratic. Draw a rectangle using *a, –a, b,* and *–b* as boundary distances from the center. The diagonals of the rectangle formed are the asymptotes of the hyperbola. The length of the transverse axis is 2a, that's the variable with the positive quadratic. The length of the conjugate axis is 2b, that the variable with a negative quadratic.

Just as we have done in with the other equations for the conic sections, we can move this curve from the origin by using

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \mathbf{1}$$

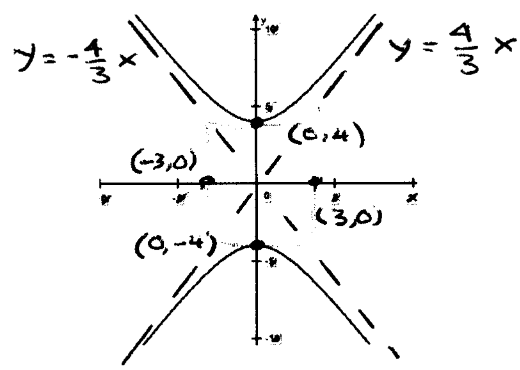
Example 1 Sketch the graph $9y^2 - 16x^2 = 144$

Since the coefficients of the quadratic terms are opposite in sign, this is a hyperbola. To write it in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form, I will divide both sides by 144.

$$\frac{9y^2}{144} - \frac{16x^2}{144} = \frac{144}{144}$$

$\frac{y^2}{16} - \frac{x^2}{9} = 1;$ $\qquad \frac{y^2}{4^2} - \frac{x^2}{3^2} = 1$ $\qquad$ Therefore, $a = 4$ and $b = 3$.

By inspection, the y-intercepts are 4 and –4. There are no x-intercepts. The asymptotes are graphs y = 4x/3 and y = –4x/3. These graphs go through the origin and the slope of the diagonals is 4/3 and –4/3.



Example 2 Find an equation of a hyperbola if the foci at (5,0) and (–5, 0), length of the conjugate axis axis is 6 and graph.

Since the conjugate axis has length 6, $2b = 6$ or $b = 3$.

The foci are located on the x-axis, in this case the transverse axis and c = 5

In a hyperbola, $\quad b^2 = c^2 - a^2$, to find a, substitute those values.

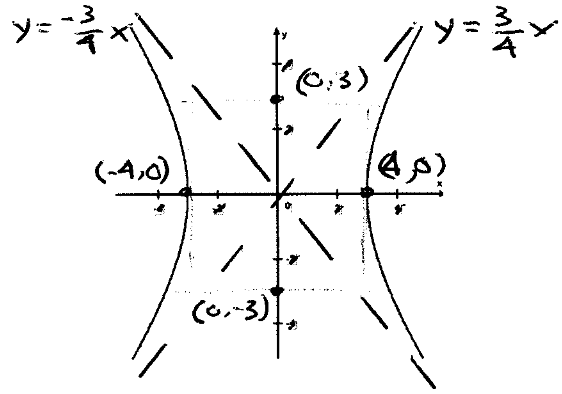
$3^2 = 5^2 - a^2$

$9 = 25 - a^2$

$a^2 = 16$

$a = 4$

Therefore the equation of the hyperbola is $\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$



Summary

In a quadratic equation, if only one variable is squared, the graph is a parabola.

if two variables are squared and the coefficients are the same, then it is a circle.

if two variables are squared and the coefficients are not equal but have the same sign, then it is an ellipse.

if two variables are squared and the coefficients have opposite signs, then it is a hyperbola

Example 1 Name the type of conic section based upon the coefficients of the quadratic term(s).

A) $5x^2 + 5y^2 + 10y - 30 = 0$

B) $4x^2 - 3y^2 + 10 = 0$

C) $x^2 + 4y + 12 = 0$

D) $3x^2 + 5y^2 + 6x - 12 = 0$

A) The coefficients of the quadratic terms are equal → circle

B) The coefficients of the quadratic terms have opposite signs →hyperbola

C) There is only one quadratic term → parabola

D) The coefficients of the quadratic terms have the same sign and are not equal →ellipse

Example 2 Find the center and radius of a circle and sketch the curve of a circle

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

We know this is a circle because it was given to us and the coefficients of the quadratic terms are the same. To determine the center and radius, I have to write the equation in $(x-h)^2 + (y-k)^2 = r^2$. To accomplish that, I will need to complete the squares.

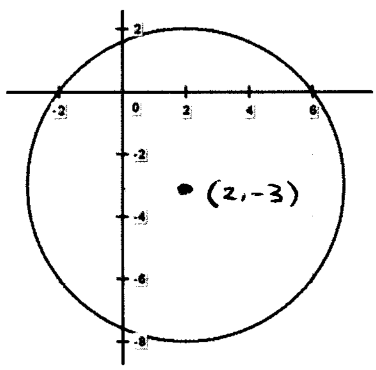
Rewriting the equation,

$$x^2 - 4x + \_\_ + y^2 + 6y + \_\_ = 12$$

$$x^2 - 4x + \underline{4} + y^2 + 6y + \underline{9} = 12 + \underline{4} + \underline{9}$$

$$(x-2)^2 + (y+3)^2 = 5^2$$

This is a circle with center (2, –3) and radius 5.



Example 3 Find an equation of a curve whose vertex is (1, 0), focus (3,0), find the equation of the directrix and sketch the curve.

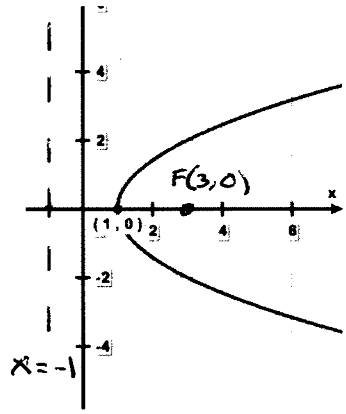
We know this will be a parabola because of the information given, focus and directrix. Because of the location of the vertex and focus, we also know this parabola will open to the right. So the y-term will be squared.

The equation will take the form, $x - h = \frac{1}{4}c(y-k)^2$

Substituting the coordinates of the vertex (1,0) and determining c = 2, the distance from the vertex to the focus gives us $x - 1 = \frac{1}{4}2(y-0)^2$

Simplifying $x - 1 = \frac{1}{2}y^2$

The equation of the directrix has to be two over from the vertex, so x = –1 is the equation of the directrix



Example 4 Find the center, foci, vertices and sketch the curve $\frac{(x-4)^2}{12}+\frac{(y-6)^2}{3}=3$

The right side of the equation must be equal to 1, so divide both sides by 3.

$$\frac{(x-4)^2}{36}+\frac{(y-6)^2}{9}=1$$

$$\frac{(x-4)^2}{6^2}+\frac{(y-6)^2}{3^2}=1$$

From this equation, we see that $a = 6$ and $b = 3$.
We also know that $b^2 = a^2 - c^2$

Substituting, $3^2 = 6^2 - c^2$

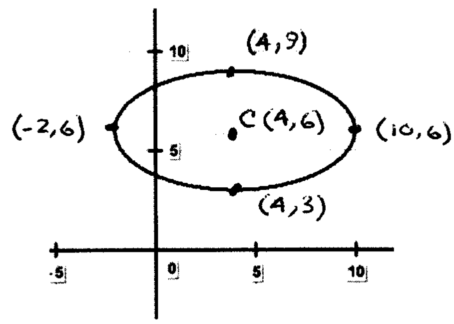
$$c^2 = \sqrt{27} \textbf{ or } c = 3\sqrt{3}$$

Now that we know the values of *a*, *b* and *c*, let's do some arithmetic to find the vertices. The center, taken right from the original equation is (4, 6). The foci are located on the major axis, since 6 > 3, that would be the x-axis. The foci are located $3\sqrt{3}$ from the center (4, 6) on the x-axis. So the foci are $(4+3\sqrt{3}, 6)$ and $(4-3\sqrt{3}, 6)$.

The vertices are located 6 units and 3 units from the center on the major and minor axes, respectively. The vertices are (4+6, 6), (4 – 6, 6), (4, 6+3), and (4, 6–3). Simplifying those ordered pairs; (10,6), (–2,6), (4, 9), and (4, +3).

All I did to find all those coordinates was add or subtract. Now on to the graph.

I graph the center and the vertices, then I'm done!



If you know the formulas and can visualize your graph, answering these questions is actually very easy because the computations are mostly arithmetic as we could see from the last example.