Derivation – Sum of Arithmetic Series

Arithmetic Sequence is a sequence in which every term after the first is obtained by adding a constant, called the common difference (d). To find the nth term of a an arithmetic sequence, we know $a_n = a_1 + (n - 1)d$

The first term is a_1, second term is $a_1 + d$, third term is $a_1 + 2d$, etc

This leads up to finding the sum of the arithmetic series, Sn, by starting with the first term and successively adding the common difference.

$$Sn = \overset{1^{st}}{a_1} + \overset{2^{nd}}{(a_1 + d)} + \overset{3^{rd}}{(a_1 + 2d)} + \ldots + \overset{n^{th}}{[a_1 + (n-1)d]}$$

We could have also started with the n^{th} term and successively subtracted the common difference, so

$$Sn = a_n + (a_n - d) + (an - 2d) + \ldots + [a_n - (n-1)d]$$

You could find the sum of the arithmetic sequence either way.

However, if you looked at that, you might see that if you added those two equations together, terms add out.

$$\begin{array}{rlllllllll} Sn = & a_1 & + & (a_1 + d) & + & (a_1 + 2d) & + \ldots + & [a_1 + (n-1)d] \\ \underline{Sn =} & \underline{a_n} & \underline{+} & \underline{(a_n - d)} & \underline{+} & \underline{(a_n - 2d)} & \underline{+ \ldots +} & \underline{[a_n - (n-1)d]} \\ 2Sn = & (a_1 + a_n) & + & (a_1 + a_n) & + & (a_1 + a_n) & + \ldots + & [a_1 + a_n] \end{array}$$

Notice all the d terms added out. So

$$2Sn = n\,(a_1 + a_n)$$

$$Sn = \frac{n(a_1 + a_n)}{2}$$

$$Sn = \frac{n}{2}(a_1 + a_n)$$

By substituting $a_n = a_1 + (n - 1)d$ into the last formula, we have

$$\text{Sn} = \frac{n}{2}\ [a_1 + a_1 + (n - 1)\text{d}]$$

Simplifying $$\text{Sn} = \frac{n}{2}\ [2\ a_1 + (n - 1)\text{d}]$$

These two formulas allow us to find the sum of an arithmetic series quickly.