Chapter 7 Polynomial Operations

Sec. 1 Polynomials; Add/Subtract

Polynomials – sounds tough enough. But, if you look at it close enough you'll notice that students have worked with polynomial expressions such as $6x^2 + 5x + 2$ since first grade. The only difference is they have letters (x's) instead of powers of ten. They have been taught that 652 means 6(100) + 5(10) + 2(1). They have been taught the six tells them how many hundreds they have, the five how many tens, and the two how many ones are 9in the number.

$$6(100) + 5(10) + 2(1) \rightarrow 6(10)^2 + 5(10) + 2 \rightarrow 6x^2 + 5x + 2$$

In the polynomial expression $6x^2 + 5x + 2$, called a ***tri***nomial because there are three terms, the six tells how many x^2's there are, the 5 tells you how many x's, and the two tells you how many ones.

The point is polynomial expressions in algebra are linked to what's referred to as *expanded notation* in grade school. It's not a new concept.

In grade school we teach the students how to add or subtract numbers using place value. Typically, we have them line up the numbers vertically so the ones digits are in a column, the tens digits are in the next column and so on, then we have them add or subtract from right to left.

In algebra, we have the students line up the polynomials the same way, we line up the numbers, the x's, and the x^2 's, then perform the operation as shown below.

$$\begin{array}{lcr} 3x^2 + 4x + 3 & \rightarrow & 343 \\ \underline{2x^2 + 3x + 5} & \rightarrow & \underline{235} \\ 5x^2 + 7x + 8 & \rightarrow & 578 \end{array}$$

Notice when adding, we added "like" terms. That is, with the numbers, we added the hundreds column to the hundreds, the tens to the tens. In algebra, we added the x^2's to the x^2's, the x's to the x's, etc. Like terms have the same variable(s) and exponents.

In algebra, the students can add the expressions from right to left as they have been taught or left to right. If the students understand place value, this could lead students to add columns of numbers more quickly without regrouping by adding numbers from left to right.

The students would have to add the hundreds column, then add tens column, and the ones column

Example 1. Add, in your head 341 + 214 + 132

You have 300 + 200 + 100, that's 600, adding the tens, we have 40 + 10 + 30 which is 680, and finally adding 1 + 4 + 2 or 7, the sum is 687.

We can add, subtract, multiply, and divide polynomials using the same procedures we learned in elementary school.

In the first grade you learned to add the ones column to the ones column, the tens to the tens, hundreds to hundreds. We use that same concept to add polynomials, we add numbers to numbers, x's to x's, and x^2's to x^2's. We call that combining like terms.

Example 2. $(3x^2 + 2x - 4) + (5x^2 - 7x - 6)$

Combining like terms, we have
$3x^2 + 5x^2$, $2x - 7x$, $-4 - 6$

$8x^2 - 5x - 10$

Subtraction of polynomials is just as easy. We will look at subtraction as adding the opposite, In other words 5 – 2 is the same as 5 + (–2). Using the reasoning, when we subtract polynomials, we will add the opposite.

Now, we will use the same trinomials and subtract rather than add. What we have to remember is to change the sign of the subtrahend, the number being subtracted, then use the addition rules for signed numbers.

Example 3. $(3x^2 + 2x - 4) - (5x^2 - 7x - 6)$

Changing the signs of the subtrahend

$(3x^2 + 2x - 4) - 5x^2 + 7x + 6)$

Grouping like terms

$3x^2 - 5x^2,\ 2x + 7x,\ -4 + 6$

$-2x^2 + 9x + 2$

Remember to change **ALL** the signs in the subtraction, then add.

All too often students do not realize a rule or procedure they are learning in algebra is nothing more than the procedure they learned in grade school. The language and notation might change, but the concepts are constant.

Simplify

1. $(3x^2 + 5x + 9) + (2x^2 + 4x + 10)$
2. $(3x^2 + 5x + 7) + (8x^2 + 4x + 3)$
3. $(5x^2 - 6x + 5) + (4x^2 - 5x - 8)$
4. $(x^2 + 7x - 9) + (10x^2 - 7x - 8)$
5. $(3x^2 + 5x + 9) - (2x^2 + 4x + 10)$
6. $(3x^2 + 5x + 7) - (8x^2 + 4x + 3)$
7. $(5x^2 - 6x + 5) - (4x^2 - 5x - 8)$
8. $(x^2 + 7x - 9) - (10x^2 - 7x - 8)$

Sec. 2 Multiply Monomial by Polynomial

To multiply a monomial by a monomial or a polynomial by a monomial, we need to remember the rules we developed for Exponentials. That is, when we multiply numbers with the same base, we add the exponents. When we divide numbers with the same base, we subtract the exponents.

Remembering those rules and applying them to monomials is a direct application of what you have done in earlier grades.

Example 1 $(x^2y^3)(x^4y^{10})$

Multiplying number with the same base
$= (x^2x^4)(y^3y^{10})$
$= x^6y^{13}$

Example 2 $(5x^2y^3)(7x^4y^{10})$

This problem is the same as the previous example with the exception of the factors of 5 and 7.

Multiplying number with the same base
$= 5(7)(x^2x^4)(y^3y^{10})$
$= 35\,x^6y^{13}$

Now to multiply or divide polynomials by a monomial, we will use the same rules again – almost. I say almost because before you can apply the rules for exponents, we will need to use the Distributive Property.

Example 3 $3x^2(4x + 5)$

$3x^2(4x) + 3x^2(5)$ Distributive Property

$12x^3 + 15x^2$ Mult same base – add exp

I can't make these problems more difficult, the best I can do is make them longer.

Example 4 $-5x(2x^2 - 3x + 4)$

$-5x(2x^2) - 5x(-3x) - 5x(4)$ Distributive Prop

$-10x^3 + 15x^2 - 20x$

Example 5

$$\frac{1}{2x^2}(8x^5 - 2x^4 + 6x^3 - 12x^2)$$

$$\frac{8x^5}{2x^2} - \frac{2x^4}{2x^2} + \frac{6x^3}{2x^2} - \frac{12x^2}{2x^2}$$

$$4x^3 - x^2 + 3x - 6$$

Sec. 3 Polynomials: Multiplication

Polynomials can be multiplied the very same way students learned to multiply multi-digit numbers in third and fourth grades.

In grade school, students are taught to line up the numbers vertically. In algebra, students typically multiply horizontally. Let's look at multiplying two 2-digit numbers and compare that to multiplying two binomials using the standard multiplication algorithm.

32	$3x + 2$
<u>21</u>	$\underline{2x + 1}$
32	$3x + 2$
<u>64</u>	$\underline{6x^2 + 4x}$
672	$6x^2 + 7x + 2$

Notice the same procedure is used in both and the digits match the coefficients.

Another way to multiply polynomials is using the Distributive Property. To use the Distributive Property, I will multiply the second polynomial (2x + 1) by 3x, then multiply it by 2. Then combine like terms.

Example 1. $(3x + 2)(2x + 1)$

$3x(2x + 1) + 2(2x + 1)$

$6x^2 + 3x + 4x + 2 = 6x^2 + 7x + 2$

Example 2. $(4x - 3)(5x + 2)$

$4x(5x + 2) - 3(5x + 2)$

$20x^2 + 8x - 15x - 6 = 20x^2 - 7x - 6$

Example 3. $(3x + 5)(2x^2 + 4x - 7)$

$3x(2x^2 + 4x - 7) + 5(2x^2 + 4x - 7)$

$6x^3 + 12x^2 + - 21x + 10x^2 + 20x - 35$

$6x^3 + 22x^2 - x - 35$

If students were to look at a number of examples, they may be able to see a pattern develop that would allow them to multiply some binomials very quickly in their head. I will write a few problems with the answers.

Example 4

$(x + 5)(x + 4) = x^2 + 9x + 20$
$(x + 10)(x + 3) = x^2 + 13x + 30$
$(x - 5)(x - 2) = x^2 - 7x + 10$
$(x - 10)(x - 5) = x^2 - 15x + 50$

Do you see a pattern?
Notice – in all those examples, the coefficient of the linear term, the number in front of the x was not written so it is understood to be ONE. When that occurs, we add the numbers to get the middle term and multiply to get the constant.

Simplify

1. $(x + 5)(x + 3)$
2. $(x + 7)(x+3)$
3. $(x + 6)(x + 2)$
4. $(x + 5)(x + 10)$
5. $(x + 8)(x - 5)$
6. $(x - 4)(x + 6)$
7. $(x - 5)(x - 3)$
8. $(x - 10)(x - 5)$

Sec. 4 **Special Products**

It is helpful to memorize certain products for ease in computation. You can probably multiply by powers of 10 in your head, multiply by 11, multiply by 25, 50 or 75 mentally also.

In algebra, you will study special products, its nothing more than mental math described by patterns.

Recognizing the patterns in these computations will help students recognize the same patterns in algebra that will help them factor algebraic expressions and solve higher degree equations. It is important that we know special products.

The special products are nothing more than patterns we will develop by multiplying polynomials.

Squaring a Binomial

Let's look at squaring a binomial using the Distributive Property we used before.

Example 1. $(3x + 4)^2$

$(3x + 4)(3x + 4)$
$9x^2 + 12x + 12x + 16 = 9x^2 + 24x + 16$

Example 2. $(5x + 3)^2$

$(5x + 3)(5x + 3)$
$25x^2 + 15x + 15x + 9 = 25x^2 + 30x + 9$

In both of those examples, notice how we added the same terms in the middle, the linear terms.

If we looked at that long enough, we might see a pattern that would allow us to multiply these binomials mentally. Let's take the coefficients out and just square a binomial with letters.

$$(a + b)(a + b) = a^2 + \underline{ab + ab} + b^2 = a^2 + 2\,ab + b^2$$

$(a + b)^2 = a^2 + 2ab + b^2$

It appears I am squaring the first term (a), squaring the last term (b), and to find the middle term, I multiply the a and b, and double it. It's important that you see this relationship, learn it, and practice squaring binomials.

Example 3. $(3x + 5)^2$ by inspection

$9x^2 + 2(3x)(5) + 25 = 9x^2 + 30x + 25$

Example 4. $(4x + 3)^2$ by inspection

$16x^2 + 2(4x)(3) + 9 = 16x^2 + 24x + 9$

I can also use this for computation.

Example 5. $23^2 = (20 + 3)^2$ by inspection

$20^2 + 2(20)(3) + 3^2$

$400 + 120 + 9 = 529$

Multiply Mentally

1. $(x + 5)^2$
2. $(x + 10)^2$
3. $(2x + 3)^2$
4. $(5x + 2)^2$
5. $(x - 10)^2$
6. $(5x - 2)^2$

Difference of 2 Squares

Let's look at the next couple of examples and see if we find a pattern that would allow us to find the product of these binomials mentally.

Example 6. $(2x + 3)(2x - 3)$

$2x(2x - 3) + 3(2x - 3)$

$4x^2 - 6x + 6x - 9$

$4x^2 - 9$

Example 7. $(3x - 5)(3x + 5)$

$3x(3x + 5) - 5(3x + 5)$
$9x^2 + 15x - 15x - 25$
$9x^2 - 25$

Notice in these two examples, the linear terms subtract out.

Let's take the coefficients out again and see if we see a pattern.

$$(a + b)(a - b) = a^2 \underline{- ab + ab} + b^2 = a^2 - b^2$$

We can see clearly that multiplication pattern, $(a + b)(a - b)$, eliminates the middle term so all we do is take the difference between the squares of the terms of the binomial.

Example 8. $(x - 4)(x + 4)$ by inspection

$x^2 - 16$

I can also use this pattern for computation.

Example 9. $(21)(19)$ by inspection

A nice number between ½ way between 19 and 21 is 20, so rewriting the problem, we have

$(20 + 1)(20 - 1) \quad = 20^2 - 1^2$
$\quad = 400 - 1 = 399$

Multiply 28 by 32 by inspection using the Difference of 2 Squares

Multiply Mentally

1. $(x + 6)(x - 6)$
2. $(x - 10)(x + 10)$
3. $(x + 3)(x - 3)$
4. $(x - 7)(x + 7)$
5. $(2x + 3)(2x - 3)$
6. $(4x - 5)(4x + 5)$

Sec 5 Binomial Expansion

In the last section, we looked at squaring a binomial and saw a pattern develop that allowed us to square binomials very quickly.

$$(a + b)^2 = a^2 + 2ab + b^2$$

Let's look at a few expansions of binomials in the form $(a + b)^n$ and see if we can find some patterns and generalizations that would allow us to expand a binomial.

$(a + b)^0 = 1$
$(a + b)^1 = a + b$
$(a + b)^2 = a^2 + 2ab + b^2$
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 +4ab^3 + b^4$

There's a few things that, with a little closer examination , stand out.

1. The expansion $(a + b)^n$ always has (n + 1) terms.
2. *b* is not a factor of the first term and *a* is not a factor of the last term
3. There is symmetry in the coefficients
4. In the first and last terms the exponent of the variable is n
5. From term to term the exponent of *a* decreases by 1 and the exponent of *b* increases by 1
6. The sum of the exponents is n
7. In any term if the exponent of *a* is multiplied by the coefficient and that product is divided by the number of that term, the result is the coefficient of the following term.

Example 1 Write the variables **without** the coefficients of $(x + y)^6$

Using statements 5 and 6 above, we have

$$x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$$

To find the coefficients, we use statement 7.

$$x^6 + x^5y + x^4y2 + x^3y^3 + x^2y^4 + xy^5 + y^6$$

To find the coefficient of the 2nd term, I multiply the exponent of the first term by its coefficient and divide by 1. So, 6 x 1 = 6, 6/1 = 6. Therefore 6 is the coefficient of the second term.

$$x^6 + 6x^5y + x^4y2 + x^3y^3 + x^2y^4 + xy^5 + y^6$$

To find the coefficient of the 3rd term, I multiply the exponent of the 2nd term by the coefficient and divide by 2. That is 5 x 6/2 = 15. Therefore the coefficient of the 3rd term is 15.

$$x^6 + 6x^5y + 15x^4y2 + x^3y^3 + x^2y^4 + xy^5 + y^6$$

To find the coefficient of the 4th term, multiply the exponent of the 3rd term by the coefficient and divide by 3 because it's the 3rd term. That is, 4 x 15/3 = 20. Therefore the coefficient of the 4th term is 20.

$$x^6 + 6x^5y + 15x^4y2 + 20x^3y^3 + x^2y^4 + xy^5 + y^6$$

From here, I can use symmetry and know the next coefficients are 15, 6 and 1 or I can continue the process I used to get the coefficients of the firsts 4 terms.

$$x^6 + 6x^5y + 15x^4y2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Example 2 Expand $(x + y)^5$

Let's make this as easy as we can using the generalizations 1 – 7. Since n = 5, I know there will be 6 terms and the exponents will decrease and increase respectively.

And, because of symmetry, we need to find half the coefficients using statement 7.

So, let's write the variables with the exponents first, then we will fill in the coefficients.

$$x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$$

To find the coefficient of the 2nd term, multiply the exponent by the coefficient of the first term and divide by 1; 5 x 1/1 = 5

$$x^5 + 5x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$$

To find the coefficient of the 3rd term, multiply the exponent by the coefficient of the 2nd term and divide by 2; 4 x 5/2 = 10

$$x^5 + 5x^4y + 10x^3y^2 + x^2y^3 + xy^4 + y^5$$

Using symmetry, we can fill in the rest of the coefficients.

$$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

If we didn't use symmetry, to find the coefficient of the 4th term, multiply the exponent by the coefficient of the 3rd term and divide by 3. 3x10/3 =10 – – as we got by symmetry.

So now that we can generalize these variables and coefficients, let's try an example that are not just simple x's and y's. The good news is, I can never make these more difficult – only longer.

Example 3 Expand $(2x + 3y)^4$

In this case, a = 2x and b = 3y, so let' write are variables with the exponents as we did before, then we will find the coefficients.

a^4		$a^3\ b$		$a^2\ b^2$		$a\ b^3$		b^4
$(2x)^4$	+	$4(2x)^3(3y)$	+	$6(2x)^2(3y)^2$	+	$4(2x)(3y)^3$	+	$(3y)^4$
$\frac{4 \times 1}{1}$	↗	$\frac{3 \times 4}{2}$	↗	$\frac{2 \times 6}{3}$	↗	$\frac{1 \times 4}{4}$	↗	

Now that we found the coefficients, we have to simplify the exponentials.

$$16x^4 \quad + \quad 4(8x^3)3y \quad + \quad 6(4x^2)(9y^2) \quad + \quad 4(2x)27y^3 \quad + \quad 81y^4$$

$$16x^4 \quad + \quad 96\,x^3y \quad + \quad 144x^2y^2 \quad + \quad 216xy^3 \quad + \quad 81y^4$$

Sec 6 Division of Polynomials

Division of polynomials is done using the same procedure that was taught in 4^{th} grade. That is, we use a trial divisor, divide, multiply, subtract, and bring down the next number in the dividend – repeat till finished.

Example 1 Divide $x^3 + x^2 - 3x - 2$ by $x + 2$

$$\begin{array}{r} x^2 - x - 1 \\ x+2\overline{)x^3 + x^2 - 3x - 2} \\ \underline{x^3 + 2x^2} \\ -x^2 - 3x \\ \underline{-x^2 - 2x} \\ -x - 2 \\ \underline{-x - 2} \end{array}$$

It's important to notice how like powers are lined up vertically throughout the problem.

Synthetic Division

Since like terms are in the same column, that will allow us to write the polynomials without writing the variables. But to do that, we MUST remember to write in a placeholder of zero for the missing term.

Recopying Example 1 with only the coefficients results in the following:

$$\begin{array}{r} 1 \ -1 - 1 \\ 1+2\overline{)1 + 1 - 3 - 2} \\ \underline{1 + 2} \\ -1 - 3 \\ \underline{-1 - 2} \\ -1 \quad -2 \\ \underline{-1 \quad -2} \end{array}$$

The way we do the division, the multiplication by 1 always subtracts out. That will allow me to leave out that step and copy all the products of (+2) on one line.

$$\begin{array}{r} 1\ -1-1 \\ 1+2\overline{)1+1-3-2} \\ \underline{1\ +2\ -2\ -2} \\ 1\ \ -1\ -1\ +0 \end{array}$$

If we continue to examine this problem, we see that we really don't need to write the coefficient of the x in the divisor.* And since the three remainders are the same as the quotient, we can omit writing the quotient and use the remainders as coefficients of the quotient polynomial.

Since we started with a cubic equation, the quotient (depressed equation) will be one degree less – a quadratic with those coefficients. The last remainder is the final remainder.

Synthetic Division – Synthetic Substitution

To make long division less cumbersome, we are going to use the same process as above with a minor change. When we divide using the standard algorithms, we divide, multiply, subtract and bring down. With a little more thought, we might notice that if we changes the sign of the constant in the divisor, rather than subtracting, we can add.

So, let's look at our original problem and write these with just coefficients and changing the sign of the constant in the divisor.

Procedure

1. Change the sign of the constant and write detached coefficients
2. Skip a space and draw a line
3. Bring down the 1st coefficient
4. Multiply the coefficient by the divisor and write the product under the next dividend coefficient and ADD
5. Repeat

Divide $x^3 + x^2 - 3x - 2$ by $x + 2$

Step 1 $\qquad -2 \,\|\, \quad 1 \quad 1 \quad -3 \quad -2$

Steps 2 and 3	–2 ||	1	1	–3	–2
		1			

Step 4	–2 ||	1	1	–3	–2
			–2		
		1	–1		

Continuing	–2 ||	1	1	–3	–2
			–2	+2	+2
		1	–1	–1	0

Write the quotient, depressed equation, with descending powers on the variable. Note that if ***n*** is the degree of the dividend, then ***n-1*** is the degree of the quotient.

- Since we are writing the dividends without the variables, it is VERY IMPORTANT that zero placeholders are inserted.
- In synthetic division, the coefficient MUST be 1.

Example 2 Use synthetic division to divide $x^3 – 2x – 21$ by $x – 3$

+3||	1	0	–2	–21
	1			

+3||	1	0	–2	–21
		3	9	21
	1	3	7	0

The quotient is $x^2 + 3x + 7$, no remainder.