Ch. 7 PROBABILITY

If someone told you the **odds** of an event **A** occurring are 3 to 5 and the probability of another event **B** occurring was 3/5, which do you think is a better bet?

Most of us would probably believe they are the same, it would not make a difference. But, in fact, they are different. Let's see how.

Probability is defined as the ratio; success to total. To write that mathematically, we have;

$$\textbf{Probability} = \frac{success}{success + failure}$$

Success plus failure is equal to the total number of things that could happen.

Odds, on the other hand, is defined as the ratio; success to failure. Writing that mathematically, we have;

$$\textbf{Odds} = \frac{\text{success}}{\text{failure}}$$

Notice, odds and probability are defined differently. Now, going back to the original question, which is a better bet, we can substitute numbers and compare.

If the odds are $3 \ to \ 5, \frac{3}{5}$, by using the definition, that means you can succeed 3 ways and fail 5 ways. In other words $s = 3$ and $f = 5$. How would that be translated to probability?

By definition, probability is $\frac{success}{success + failure}$. By plugging in the numbers, we have

$$\frac{3}{3+5} = \frac{3}{8}.$$

So if the odds are 3 to 5 of an event occurring, then the probability of that event occurring is 3/8

Now, which is the better bet, **A** or **B**? Hopefully, you would now say **B**. In **A** you have three chances out of eight to win, in **B** you have three chances out of five.

If I know the probability of an event, I can determine the odds associated with the event.

If the probability is $\frac{3}{5}$, by using the definition $\frac{success}{success + failure}$, that means you can succeed 3 ways out of a total $(s + f)$ 5 ways. We already said $s = 3$, so if $3 + f = 5$, what's the value of *f*. You've got it, 2. So $f = 2$. Now writing the odds we have

$\frac{success}{failure} = \frac{3}{2}$ The odds are three to two.

Piece of cake, don't you think?

Converting Probability – Odds

To convert probability to odds and odds to probability, we can formalize what we just did using this procedure.

1. Write the information given with its respective ratio
2. Determine the value of s and f
3. Write the ratio for the information you are looking for
4. Plug in the values of s and f

Let's try one.

Example 1 Find the odds of an event if the probability is given by $3/4$

1. $P = \frac{s}{s + f} \rightarrow \frac{3}{4}$.
2. In our case *s=3*. The denominator is 4 and that is equal to $s + f$
   If $s = 3$, then *f* would have to equal 1. Knowing that *s* is 3 and *f* is 1, all I had to do is plug these numbers into the definition for odds.

3. **Odds** $= \frac{Success}{failure}$ 4. **Odds** $= \frac{3}{1}$

***Example* 2** Find the probability of an event if the odds are given by 2 to 3.

1. $\text{Odds} = \frac{\text{success}}{\text{failure}} \rightarrow \frac{2}{3}$.
2. s = 2 and f = 3
3. $P = \frac{success}{success + failure}$.
4. $\Pr obability = \frac{2}{2+3} = \frac{2}{5}$

Finding Probabilities of Events

The concept of probability is very simple, a ratio, the number of ways you can succeed over the total number of ways something can happen.

Before we go too far, let's define some terms.

Experiment – *i*s an activity under consideration, such as flipping a coin.

Outcome – is one of the possible things that can happen in an experiment.

Sample space - the set of all possible outcomes in an experiment.

Event - a list of specific outcomes within a sample space – a subset of the sample space.

Mutually exclusive events – events are mutually exclusive if the occurrence of one event prevents the occurrence of the other. $A \cap B = \varnothing$

Independent events – two events, A & B, are independent if the occurrence of A has no effect on the probability of B.

***Example* 3** Now, let's say we flip a coin, we'll call that the experiment. The set of all possible outcomes – the sample space consists of heads or tails.

If I wanted to know the probability of getting a head, then I use the definition of probability.

$$\text{Probability} = \frac{success}{sucess + failure}$$

I can succeed one way (getting a head on the toss), over the total number of things that can happen (sample space, success + failure).

$$\text{Probability (heads)} = \frac{1}{2}$$

Pretty easy, don't you think?

Rules of Probability

R1. The probability of an impossible event is zero.
R2. The probability of an event that is certain to occur is one.
R3. The probability of an event must be between zero and one.
R4. The sum of all the possible outcomes for any sample space must be one.

Example 4 A card is drawn from an ordinary deck of cards, what is the probability that it is an ace?

In a deck of cards, there are 52 cards. There are 4 aces, so there are 4 ways you can succeed. Therefore, the Probability (ace) = 4/52

This is important, the probability of an event A is equal to the sum of the probabilities of all the outcomes in set A. So, a different approach to the last problem, since the probability of drawing any of the four aces is 1/52, then

$$\text{Probability (ace)} = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{4}{52}$$

While the concept of probability is a simple one, the difficulty one runs into is knowing ***all*** the possible things (outcomes in the sample space) that can happen.

If I were to ask you what the probability of picking your name out of a hat that contained ten names, you would quickly see that P = 1/10, because you knew there were only ten names.

If we are familiar with certain activities, that's very straightforward. However, sometimes knowing all the possible outcomes is not as easy as it sounds.

Example 5 Put all the names of the 50 states in a hat and find the probability of picking a state that begins with the letter A.

If you didn't know all the states, you might not get the correct answer. However, if you know there are 50 states and there are 4 states that begin with A, then finding the probability is easy, it's 4/50.

The states are Alabama, Alaska, Arkansas, and Arizona. Each state has a probability of 1/52 to be picked out of the hat.

$\frac{1}{50} + \frac{1}{50} + \frac{1}{50} + \frac{1}{50} = \frac{4}{50}$ That leads us to another rule

***Example* 6** If I flip two coins, a quarter and a dime, what's the probability of getting two heads? What are the possible outcomes?

I'm going to use capital letters for the quarter, small letters for the dime and list them.

Here's the list: *Hh*, *Ht*, *Th*, *Tt*.

R5. If events, A & B, are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

There are four outcomes. Now, answer the question, what is the probability of tossing two coins and have them both come up heads? From my list I see that happens once out of four times.

$$\text{Probability (both heads)} = \frac{1}{4}$$

Again, if we know how large the sample space is, then finding the probability is as simple as plugging in the number of ways you can succeed over the total number of outcomes. Did you know there were four different things that could happen if you flipped two coins? How about flipping three coins, how many different things could happen?

The problem then, is finding the total number of ways something can happen. Finding the number of outcomes in a sample space.

Example 7 There are 4 queens and 13 hearts in a deck of 52 cards, find the probability of picking a queen or a heart. P(q U h)

There are 4 queens in a deck, 13 hearts, and one queen of hearts is counted twice, so I need to subtract that.

we can write P(q U h) = P(q) + P(h) – P(q and h)
= 4/52 + 13/52 –1/52 = 16/52 = 4/13

Another person could gave reasoned there were 13 hearts in the deck of cards, plus 3 queens they did not already count or 16/52.

That leads us to the next two rules.

R6. The probability of two events A and B that have common intersection, P(A and B) = number of (A and B)/ number in sample space.

R7. If two events, A & B, are not mutually exclusive, then P(A or B) = P(A) + P(B) – P(A and B)

.

Example 8 If event X represents spinning an even number on a circular wheel with 5 numbers with equal area and Y represents spinning a number less than 4, find the P(X and Y)

X = {2, 4} Y = { 1, 2, 3). They have "2" in common

$$P(X \text{ and } Y) = \frac{n(X \text{ and } Y)}{n(S)} = \frac{1}{5}$$

Example 9 Using the same information from example 8, what's the probability of X **or** Y?

It would seem the P(X U Y) = P(X) + P(Y);

X = {2, 4} Y = { 1, 2, 3). Notice, they have "2" in common.

That results in counting "2" twice. So when we have P(X or Y) we have to determine if there is an intersection. If there is, we will count those outcomes a multiple number of times. To avoid that, we need to subtract out that duplication/intersection so we don't have double counting.

P(X or Y) = P(X U Y) = P(X) + P(Y) – P(X and Y)

= 2/5 + 3/5 – 1/5 = 4/5

Example 10 There are 5 pieces of paper in a hat numbered 1, 2, 3, 4, 5. What's the probability of drawing the paper with the number 3?

That's easy enough, P(3) =1/5.

What if I wanted to know what the probability of drawing a number that is **<u>not</u>** 3, called the complement, written ~3 or 3' or 3^0? One way to find that is to find the probabilities of picking a 1, 2, 4, and 5, then add them together. A better way, using logic, is to just subtract the P(3) from 1.

That leads us to our next rule

R8. **P(~A) = 1 – P(A)**

Finding the Sample Space – Listing

One way to find everything in a Sample Space is to list them.

Example 11 If I asked what are all the possible outcomes when I flip a coin and roll a die.

By listing, I have
H1, H2, H3, H4, H5, H6
T1, T2, T3, T4, T5, T6

We can see there are 12 things that could happen. Now, if I asked the probability of getting a head and a 5, from the list you can see I can succeed one way out of a total of 12.

Therefore, the $P(H,5) = 1/12$

Again, the concept of probability is quite simple. If you know all the possible outcomes, then all you do is put the number of ways you can succeed over the total number of things that could happen. Piece of cake!

Example 12 In a committee of 5 people; A, B, C, D, E, what is the probability that A will be chosen president and C vice president, if they were chosen at random.

Again, the question comes down to; what is the sample space? I could try to list those as I did before or I could use a more systematic method.

The probability of choosing A, president followed by C as vice president is 1/20. Try to list all those possibilities. You can see trying to list all those is a pain the rear, we might see a need for a better approach than just trying to list all the possible outcomes.

Let's look at some problems where finding the sample space is pretty straight forward.

1. Find the probability of drawing the "2 of Diamonds" from a deck of 52 cards.
2. Find the probability of drawing a King from an ordinary deck of cards.
3. Each letter of the alphabet is written on a separate sheet of paper and placed in a box, then one piece of paper is drawn randomly, what is the probability the paper drawn is a vowel?
4. What is the probability of picking a month of the year at random that begins with the letter "J"?
5. If student council has 15 members, 7 boys and 8 girls, what is the probability that a boy will be selected if someone has to be chosen at random to run an errand?
6. A golf course has eighteen holes, if 5 of them are considered water holes, what is the probability that a person playing on that course is not on a water hole?
7. The lunch wagon has 4 tuna sandwiches, 5 egg salad, 2 ham, and three turkey sandwiches. If you reached in at random to select a sandwich, what is the probability of selecting a tuna or egg salad sandwich?
8. If the probability that it will rain tomorrow is .6, what are the odds it will rain?
9. If the odds in favor of the Red Sox winning the series is 2 to 5, what is the probability they will win the series?

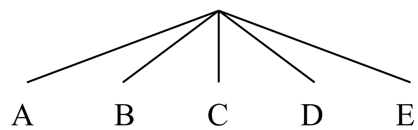
Sometimes finding all the outcomes is not as easy as above. So, we will look at a more systematic approach like we discussed in Example 12 – called a Tree Diagram.

Tree Diagrams

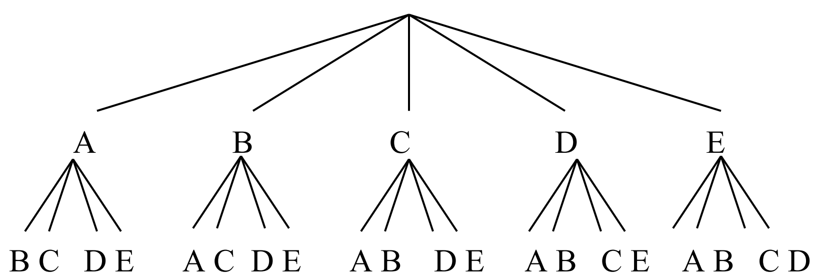
A tree diagram is just a systematic way of listing all the possible outcomes.

A tree diagram is a diagram that starts out with a point with branches that identify what can happen on the first stage of an experiment. For the second stage, more branches to identify what can happen in the second stage are based on what has already happened in the first stage. This process is continued until we have all possible outcomes.

Looking at the last example 12. To draw this tree diagram, I first list all the ways I could select the president of the committee. The president could be **A, B, C, D** or **E.**



After I list all of those, then I list the ways I could pick the vice president, given the previous selection of the president. So, if A was chosen president, then B, C, D, or E could be vice president. Continuing that process for the other branches, if B was chosen president, then A, C, D, or E could be vice president.



5 outcomes

4 outcomes

Now, reading down the branches gives me all the possible things that could happen. I could have had AB, AC, AD, or AE. If B was chosen as president, I could have BA, BC, BD, or BE.   If C was chosen first, then the outcomes could be CA, CB, CD, or CE. With D being chosen first, we'd have DA, DB, DC, or DE. And if E was chosen present, then the results could have been EA, EB, EC, or ED.

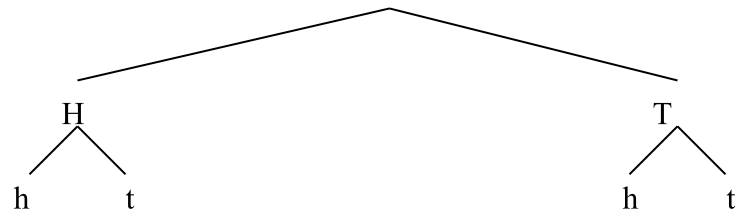
That's a total of 20 things that could happen, the Sample Space. Neat stuff, don't you think? I have only one AC, therefore the probability is 1/20.

The diagram is just a systematic and visual way of identifying all the outcomes.

Example 13 Make a tree diagram for tossing the quarter and the dime we did on the last page.

Quarter — 2 outcomes

Dime — 2 outcomes



The capital letters represent the quarters, the small letters represent the dimes. Reading down the tree diagram, we have Hh, Ht, Th, and TT. There are 4 things that could happen.

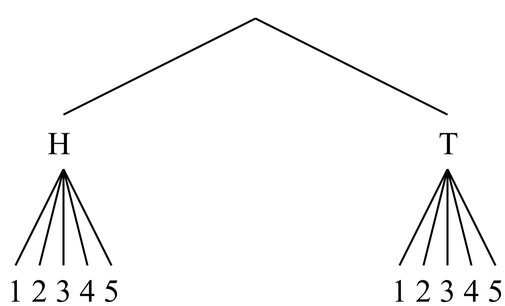
The tree diagram is a very important tool in solving beginning probability problems. To use it, you list all the ways the first event could happen. Under each of those branches, you list all the ways the next event can occur given the first one has already occurred.

N.B. I placed the number of outcomes for each stage to the right of the diagrams, I will refer to them later.

Example 14 Draw a tree diagram to show all the possible outcomes of tossing a coin, then throwing a five-sided die.

What can happen when you toss a coin? You either get a head or a tail, that's your first branch. If you get a head, you toss the die, what could happen? You get a 1, 2, 3, 4, or 5, What would happen if you would have gotten a tail? You would have thrown the die and it would have turned up a 1, 2, 3, 4, or 5.

Let's draw the tree diagram.



2 outcomes

5 outcomes

The total number of outcomes, the sample space, is 12. Flipping a coin and rolling a die can may result in any one of the following; H1, H2, H3, H4, H5, T1, T2, T3, T4, T5.

Draw a tree diagram for each of the following problems.

1. Draw a tree diagram and find the probability of having 2 boys in a family of 3 children?
2. On a three question true-false test, draw a tree diagram and find the probability of getting 100% if all the answers were chosen at random.
3. A sports car can be ordered in 5 colors and two types of transmissions, manual and automatic, draw a tree diagram to show how many types of cars can be ordered.
4. Three coins are tossed, what are the possible outcomes?

If you looked at the numbers for each of the outcomes in the tree diagrams, a pattern kind of jumps out at you. The product of those numbers is the total number of outcomes.

By looking at the numbers assigned to the branches of the previous 3 tree diagrams, we see that 5x4 equals the 20 possible outcomes; that 2x2 equals the 4 possible outcomes in that example, and 2x5 equals the 10 possible outcomes in the last example. This is important, so if you didn't go back and look at the tree diagrams and the numbers associated with the branches, do it now! Seeing that sets us up for a very important principle that will save you a lot of time.

Fundamental Counting Principle – if an event **M** can happen in ***m*** ways, and after that occurs another event **N** can happen in ***n*** ways, then event **M** followed by event **N** can happen in ***mxn*** ways.

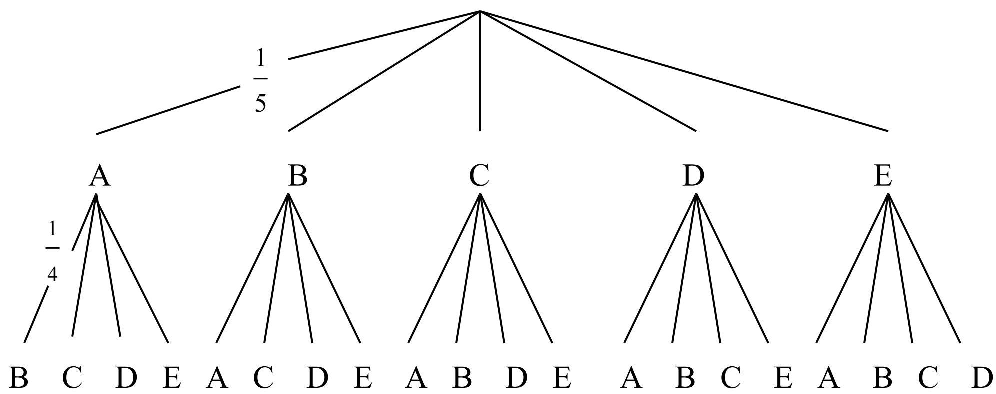
Up to this point, all the outcomes we have worked with have been equally likely. In other words, we assumed that the probability of a coin landing on heads was ½. We assumed rolling a die and landing on"3" was 1/6. That doesn't always have to be the case.

Assigning Probabilities to Each Branch of a Tree Diagram

If we were to go back and look at the tree diagrams we already drew and made a few more, we might see a pattern develop. The pattern that might jump out at you is the total number of outcomes in a sample space could be determined by multiplying the number of outcomes in each stage of an experiment. For instance, when we flipped the coin, we saw two outcomes. When we rolled the die, 6 things could happen. By drawing the tree diagram, we found there were 12 different outcomes – or 2 x 6.

Using that observation gives us some more good news. If we were to assign probabilities to each branch of the tree diagram using the number of outcomes, we might see something interesting.

Let's copy the tree with the president and vice president and assign probabilities to each branch.



Because there are 5 ways to choose the president, the probability of each of the top branches is 1/5. I pick the vice president **<u>after</u>** the president is chosen, so each of those branches is 1/4.

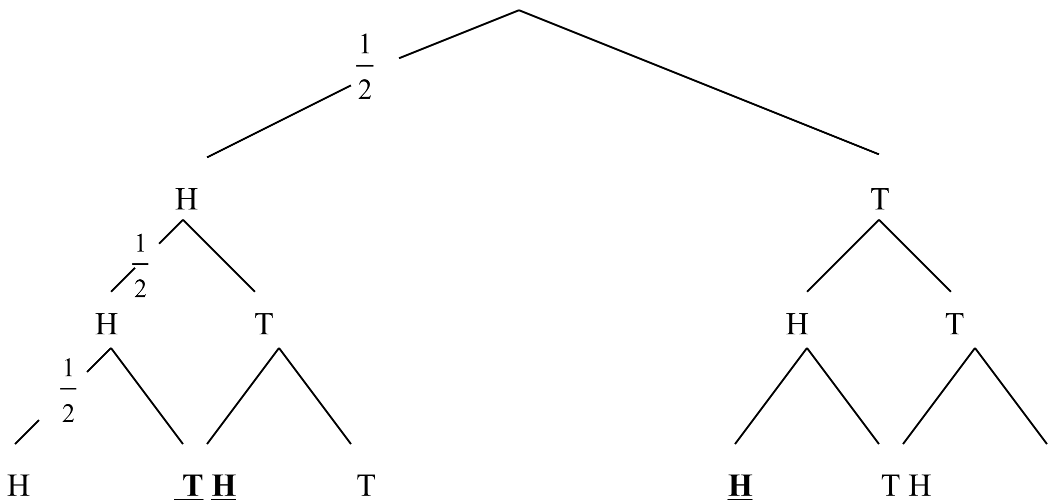
Notice, if we multiply down the branches to get A followed by B, we get the probability of that particular outcome. So, the probability of having A president and B vice president is $\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{20}$ That leads us to this very important note.

We can see from the Fundamental Counting Principle
For all multistage experiments, the probability of an outcome along any path of a tree diagram is equal to the product of all the probabilities along the path.

Let's check that out with another example.

Example 1 Tossing three coins, a quarter, dime and nickel, what's the probability of getting 2 heads and a tail?

Drawing the tree diagram and assigning the probabilities to each branch, we have



I underlined the branches that led to 2 heads and one tail, Looking at that and using the rule for all multistage experiments that the probability of an outcome along a path is equal to the product of the probabilities along the path that lead to that outcome, we can quickly see the probability of getting 2 heads and one tail is. I've underlined each outcome that gives us two heads and a tail. The probability of that outcome is found by multiplying down the branches.

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{1}{8}$$

The question we were asked was to find the probability of getting 2 heads and a tail. So, what we have to do is look at the tree diagram to determine which outcomes satisfy that event occurring. There are three outcomes that result in 2 heads and a tail; HHT, HTH, THH – they are underlined above. The probability of each of those outcomes is 1/8. So, the probability of the event is equal to the sum of the outcomes that make up the event. That is 3/8

Isn't this a blast!

What's nice about math is I cannot make these problems more difficult, I can only make them longer.

Modified Tree Diagrams & Outcomes Not Equally Likely

Up to this point, the outcomes in the tree diagrams have been equally likely. That is each branch on each stage had the same probability assighed to it. That doesn't always happen, what a surprise! But, the good news is that doesn't change what we know, we use the same algorithm as below.

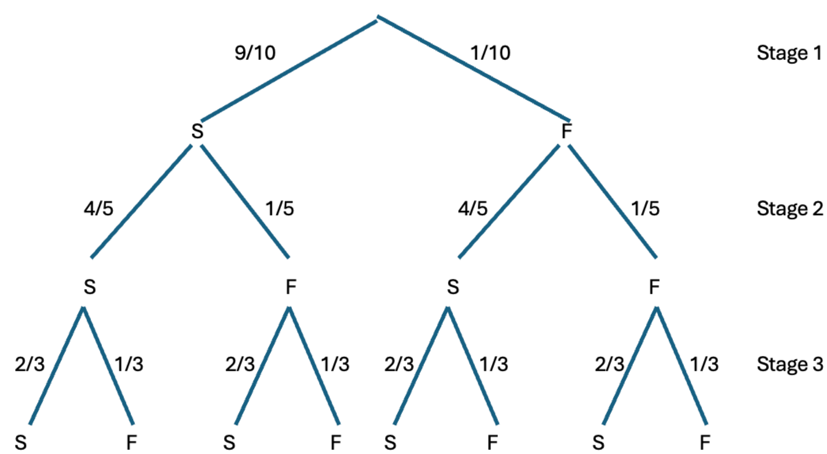
So, let's look at a problem where the outcomes are NOT equally likely.

To find probabilities using a tree diagram:

1. Make the tree diagram.
2. Label each branch's probability
3. Multiply down the branches to find the probability of the desired outcome.

To find the probability of the event, you take the sum of the outcomes that make up the event. Let's look at this problem where a tree diagram is helpful.

Ex.. Suppose a three-stage rocket is launched. The probability for failure during stage one is 1/10. At stage two, the probability of failure is 1/5 and at stage three its 1/3. What is the probability of a successful flight? (Hint – no failures)

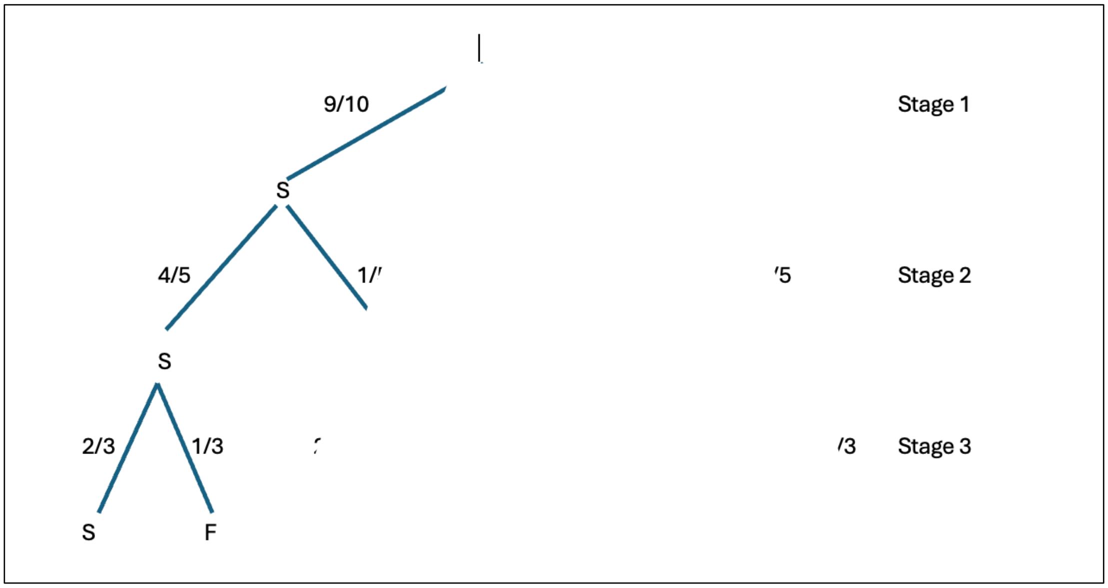


A successful flight means there can be no failures. So, following the branch that has success, followed by success, followed by success, we have:

$$\frac{9}{10} x \frac{4}{5} x \frac{2}{3} = \frac{72}{150} = \frac{12}{25}$$

If we looked at that problem more closely, we might notice that we only used the part of the tree diagram in which we had an interest. The rest of the diagram had nothing to do with the problem.

Knowing that, would have made constructing the tree diagram easier.



This is a modified tree diagram, drawn based on what we were asked to determine – a successful flight. Note we didn't need any part of the diagram that resulted in a failure.

Try a couple of these using modified tree diagrams.

1. On a certain street there are three traffic lights. At any given time, the probability that a light is green is 0.3. What is the probability that a person will hit all three lights when they are green?

2. A fair coin is tossed and a six-sided die was thrown, what is the probability that it results in a head and an even number?

3. On a three question True-False test, what is the probability of getting all the answers correct if the student guessed the answers?

4. If A, B, C, D and E are vying to go on a school trip and their names are put in a hat, what is the probability of B and D being picked from the hat?

Working with tree diagrams can become very cumbersome, particularly by the third or fourth stage of an experiment – especially if we cannot modify them. But even modifying tree diagrams can be cumbersome. Because of this, we will soon discover other ways of determining probabilities using different counting methods.

*** If you have not covered the chapter on Counting Methods, stop here and go to that chapter. In that chapter you will study the fundamental counting principle, permutations, and combinations. If you have covered the counting methods, we can go on.

PROBABILITY

Student should be able to:

- Find the probability of a simple event by counting
- Find the odds of an event.
- Find the probability of a multi-stage experiment using a tree diagram

<u>Problem Set</u>

1. What is the probability of picking a month of the year at random that begins with he letter “J”?

2. Find the probability of selecting a state at random that begins with the letter “A”.

3. In the set $S = \{1,2,3,4,5,6,7,8,9\}$, find the probability of selecting a prime number if a number was chosen at random?

4. If a card is selected at random from an ordinary deck of playing cards, what is the probability that the card is a king?

5. A drawer contains six black socks, four brown socks, and two green socks. Suppose one sock is drawn at random and it is equally likely that any of the socks will be picked, find the probability that the sock is either black or brown.

6. If the student council has 15 members, 7 boys and 8 girls, what is the probability that a boy will be selected president, if the president is chosen at random?

7. What is the probability of selecting a number that is divisible by 4 from the following set; $\{3,6,9,12,15,18,21,24,27,30,33,36\}$

8. In problem 7, what is the probability of NOT picking a number divisible by 4?

9. A golf course has eighteen holes, if 5 of them are water holes, what is the probability that a person playing golf is on a water hole?

10. The lunch wagon has 4 tuna sandwiches, 5 egg salad, 2 harm, and 3 turkey sandwiches. If you reached in at random to select a sandwich, what is the probability of selecting tuna or egg salad sandwich?

11. A coin is flipped twice, what is the probability that 2 heads come up?

12. A coin is tossed in the air, then a die is rolled – both are fair. What is the probability of getting a head and a 5? What is the probability of getting a tail and an even number?

13. If Bob's batting average is .300, what is the probability that he will get two hits in a row?

14. If Bob's batting average is .300 and he get to go to bat three times, what is the probability he will get two hits? At least two hits?

15. A study of people with an MBA degree shows that it is reasonable to assign a probability of .800 that such a person will have an annual salary in excess of $30,000. Find the probability that such a person will earn $30,000 or less?

16. If the odds in favor of the Red Sox winning the series is 2 to 5, find the probability that they will win?

17. If the probability that it will rain tomorrow is .6, what are the odds it will NOT rain?

18. If the probability that it will rain over the next two days is .6 for each day, what is the probability it will rain both days?

19. On a 3-question multiple choice test, the probability that you will guess the correct answer is .2, what is the probability that you will get 2 out of 3 correct?

20. In a class of 10 students; three students earned an A, five students earned a B, and two students earned a C. If three students were chosen at random, determine the probability that all three students earned an A. Determine the probability that no students earned an A. (Hint – might be a good time to recall the complement)

Name______________________________

Probability

Date_______________________________

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Definitions

1.*** Probability

2.*** Odds

2.*** Mutially Exclusive Events

3.*** Fundamental Counting Principle

4.*** Procedure for finding probability of an event using s tree diagram.

5.*** Event

6.*** Tree Diagram

7.*** Permutation and write formula

8.** A fair coin is tossed 5 times . What is the probability of getting all 5 heads?

9.** A couple plans to have three children, what is the probability of having at least one girl?

10**. A teacher prepared a 4 item true-false test, how many ways can the test be answered?

11.** A cast for a school play is to be selected from 7 eligible girls and 9 eligible boys. The play requires 4 girls and 3 boys. In how many ways can the players be selected?

12.** Sharon has 3 skirts, 4 blouses and 2 pairs of shoes in her closet, assuming they can be made to match, how many different outfits does she have available to wear?

13.** How many different 3-person committees can be formed from 5 people?

14.** If the probability in favor of the Red Sox winning the pennant is 8/15, what are the odds that they win?

15.** In how many ways can a party of 4 men and 4 women be seated in a row if the men and women are to occupy alternate seats?

16.** Find the probability of rolling a sum of a 7 or 11 when rolling a fair pair of dice.

17.** At a local college, 60% of the students are undergraduates, 45% of the students are male, and 40% of the undergraduates are male. What is the probability a randomaly chosen student is either male or and undergraduate?

18.** The batting average of a baseball player is .300. What is the probability that he will hit two out of three times at bat?

19.*ACT/SAT What is the probability of winning Mega Millions if you have to match 5 numbers out of 70, followed by one number out of 25?

20.*ACT/SAT An assembly line has two inspectors. The probability that the first inspector will miss a defective item is 0.05. If the defector item passes the first inspector, the probability the second inspector will miss it is 0.01. What is the probability that a defective item will pass by both inspectors?

21.*** Provide contact information for a parent/guardian, home or cell phone, email adress or address (CHP)